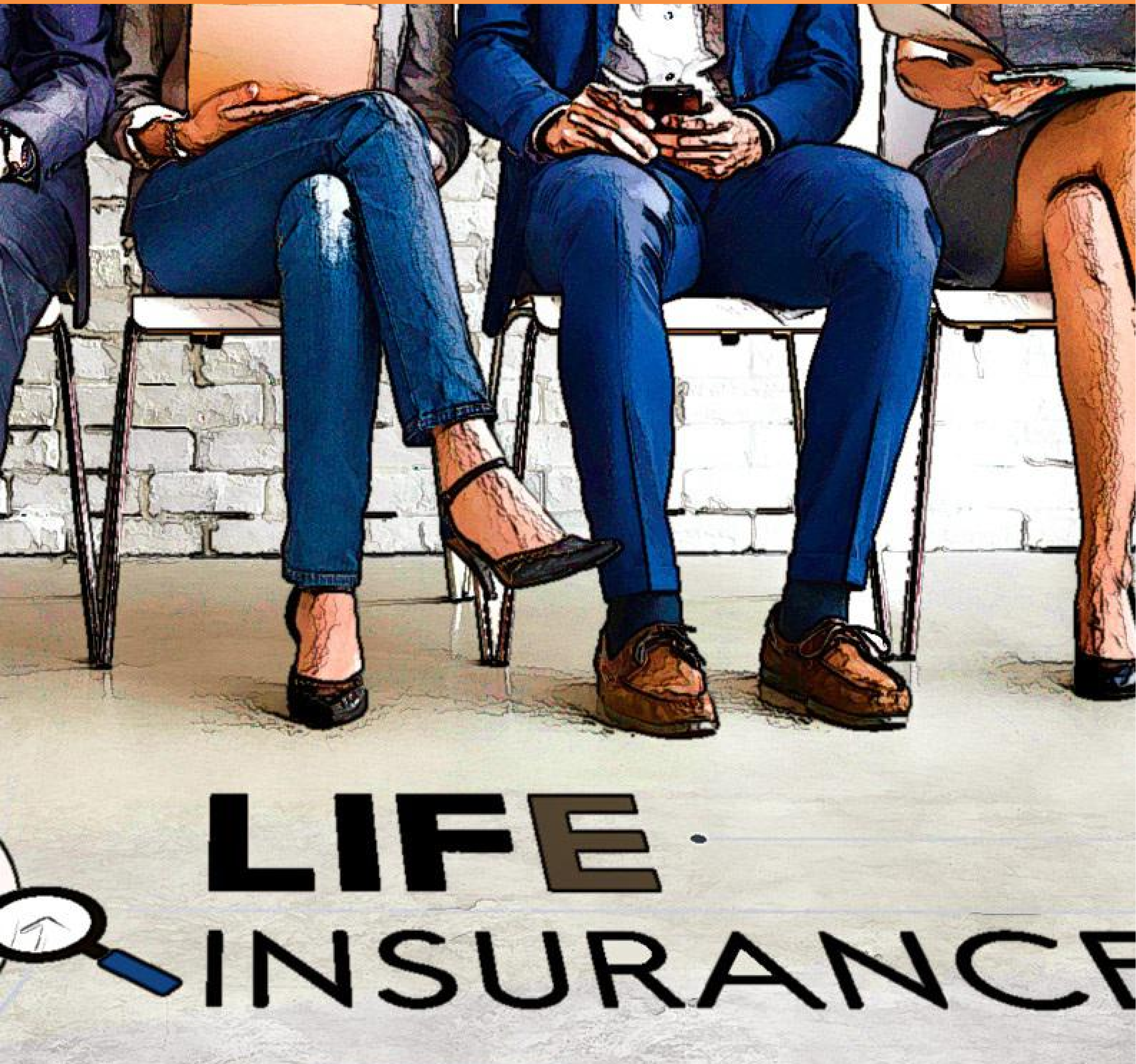



# Life Insurance

Abdul H. Rahman; Dick H. Harryvan

bookboon
The eBook company

DICK H. HARRYVAN AND ABDUL H. RAHMAN

# LIFE INSURANCE

Life Insurance
1st edition

ISBN 978-87-403-2035-0

# CONTENTS

# ABOUT THE AUTHORS

Drs. Dick Harryvan is Vice Chairman of the Supervisory Board of NN Group in the Netherlands an insurance and asset management group operating in 18 countries. He is a former Executive Board Member of ING Group and CEO of ING Direct globally. After obtaining a Masters degree in business economics from Erasmus University, an international career in insurance and retail banking followed. For Nationale Nederlanden Group, he spent nine years in various functions in the US and Canadian business units. The next four years at the international division included amongst others responsibility for setting up new life insurance greenfields in Hungary and Czech Republic after the Wall fell. He then moved to ING Bank to develop retail banking internationally for ING Group. In a three-year incubator period, various concepts were tested with responsibility for the ING Direct pilot in Canada. After the initial success in Canada this concept was rolled out to nine countries growing to 24 million customers in the space of 10 years. Since retirement from the ING Executive Board in 2010 several supervisory board positions for ING Group subsidiaries followed. As one of the three founders, he set up the Retail Banking Academy in London in 2012 with the aim of making retail banking a recognised profession. Other positions include membership of the Board of the Dutch automobile association ANWB and partner and investment committee member at OGC a private equity fund investing in fintech startups.

I dedicate this book my wife Corrie without whom my career would not have been possible.

Rotterdam, The Netherlands, 2017.

Professor Dr. Abdul H. Rahman holds academic degrees that include MSc. (Mathematics); M.A. (Economics); and Ph.D. (Financial Economics). Over his career, he held positions as Full Professor of international economics and finance and Telfer Teaching Fellow at the Telfer School of Management, University of Ottawa, Canada; Associate Dean at what is now the John Molson School of Business, Concordia University in Montreal, Canada; Academic Director and Founding Director of the Retail Banking Academy, London, United Kingdom and Chairperson of Mathematics and Physics, Dawson College, Montreal, Canada.

He has published in several prestigious refereed academic journals including the Journal of Banking and Finance, Journal of Asset Management, Journal of International Financial Markets, Institutions and Money, Journal of Financial and Quantitative Analysis, Journal of Futures Markets and Review of Economics and Statistics.

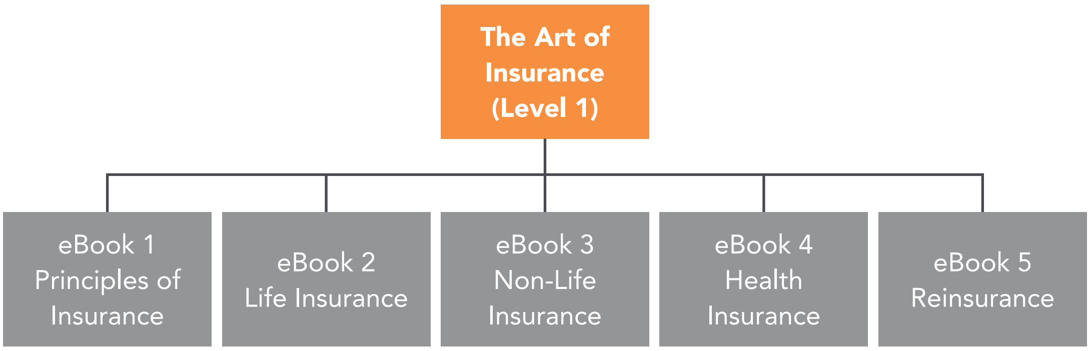
Dr. Rahman has consulted for global financial institutions in banking, asset management and insurance and currently facilitate high-level seminars for senior management professionals in retail and corporate banking, life and non-life insurance and institutional asset management.

I dedicate this eBook to my wife Ruth and daughters Sara and Lisa for their unfailing support of my professional pursuits.

Montreal, Canada, 2017.

# PREFACE

The ART OF INSURANCE series is structured according to three levels – I, II and III, each with increasing knowledge and understanding of insurance as a business. The architecture of The Art of Insurance (Level I) is schematically summarised as follows:




Each eBook in Level I is developed independently and is self-contained. Hence, the reader is not required to complete the five eBooks sequentially. Consequently, some information presented in the first eBook entitled the *Principles of Insurance* will be repeated so that there is a continuous flow in our presentation.

In addition, our approach focuses on the 'art' rather than the 'science' of insurance where we emphasise principles, concepts and intuition rather than mathematical proofs of complex theorems. While we recognise that the insurance business is founded on a high level of sophistication in probability and statistics, we avoid an overbearing level of jargon and a misplaced reliance on complex equations and formulae. Throughout this eBook, we present required actuarial notation in separate boxes while providing intuitive and logical explanations in the body of the chapter through several illustrative examples and case studies.

This eBook entitled the *Life Insurance* comprises five chapters.

Chapter 1 presents a description of the attributes of life insurance contracts and life annuities ranging from endowment insurance, term life and whole life insurance to corresponding life annuities that may be classified as contracts with contingent payments. We also introduce variable annuities (also called unit-linked insurance) with guarantees and options as well as a review of annuities (due and immediate), deferred annuities and life annuities with period certain. In this chapter, we also introduce actuarial concepts and notation including complete and curtate lifetime random variables, survival and mortality probabilities and force of mortality.

Chapter 2 considers the information content of life tables as well as the classical De Moivre mortality model and its resulting survival function. We choose the illustrative life table (ILT model) as the basis for our examples and illustrations of the key concepts and principles and to demonstrate common actuarial symbols and terminology. This is partly because the ILT model is preferred by the Exam Committee of the Society of Actuaries for candidates taking the accredited examination on 'Models for Life Contingencies'.

Chapter 3 deals with premium calculation principles for life annuities including whole life annuities temporary life annuities, deferred annuities and period certain whole life annuities. We also identify important relationships between the actuarial present value of various life annuities and consider the case of payments made to the annuitant $m$ times during a year. The Woolhouse 2-term formula is useful is this regard.

Chapter 4 considers the premium calculation principles for life insurance and derives important relationships between life insurance and life annuities. In particular, we demonstrate a link between life annuities and life insurance that leads to simple calculation procedures for single and annual premiums for life insurance.

Chapter 5 is a final chapter comprising a list of references used in this eBook.

# 1 LIFE INSURANCE AND LIFE ANNUITIES

## 1.1 INTRODUCTION

As stated in the Preface to this eBook, our approach focuses on the 'art' rather than the 'science' of insurance where we emphasise principles, concepts and intuition rather than mathematical proofs of complex theorems. However, achieving an *intuitive* understanding of life insurance and life annuities depends on mastering two key principles of finance and probability theory.

The first principle is a statement concerning the process of discounting future cash flows to the present time. For example, in life insurance the policyholder typically pays regular premiums over the term of the contract starting from the date of policy issue. The issuer pays a benefit contingent on the death or survival of the insured at some point in the future. These two streams of cash flows are not synchronous. They can only be compared at **a** point in time and discounting cash flows to the present time is crucial.

This principle amongst others, is presented in the first eBook entitled *Principles of Insurance,* and a brief review is found in Box 1.3 in this chapter.

The second principle concerns a challenging issue faced by life insurance actuaries and may be stated as a question: for an individual of age $x$ years, what is his/her expected remaining lifetime? The answer to this question helps to determine appropriate premiums for life insurance and life annuities.

We consider this principle under the title of *complete expected life* in section 2.1.5 of Chapter 2.

The central focus of this chapter is a description of the fundamental characteristics of life insurance and life annuity contracts that facilitates premium calculation methods in chapters 3 and 4. The insured event for life contracts – whether insurance or annuities – is typically the **death** or **survival** of an individual within a specified period of time. For example, a term life insurance is based on the death of the insured within a fixed number of years while a pure endowment insurance pays a benefit at the end of a specified term based on the survival of the individual.

For life insurance, the policyholder **pays** the insurer regular premiums and receives a benefit if the insured dies within the specified term. For life annuities, payments are made in reverse whereby the individual **receive**s regular payments from the insurer in exchange for a single premium paid to the insurance company at the date of policy issue. The payments cease upon death of the individual.

To facilitate the presentation of the issues in this chapter, we define the parties to a life insurance contract.

A **policyholder** is the individual who has a contractual obligation to make premium payments to the insurer. The **insured** is the individual who is the source of the mortality risk that the insurer bears. In other words, the expected future lifetime of the insured determines the premium payments the policyholder makes to the insurer.

*For simplicity and by common practice, we* ***assume*** *that the policyholder and the insured are the same individual. That is, the policyholder purchases life insurance and makes premium payments based on his/her own lifetime.*

Finally, the **beneficiary** receives the benefit payment (also called the **sum insured)** if the insured event occurs. In the case of death, the beneficiary may be the policyholder's estate.

Figure 1.1 illustrates this contractual relationship:

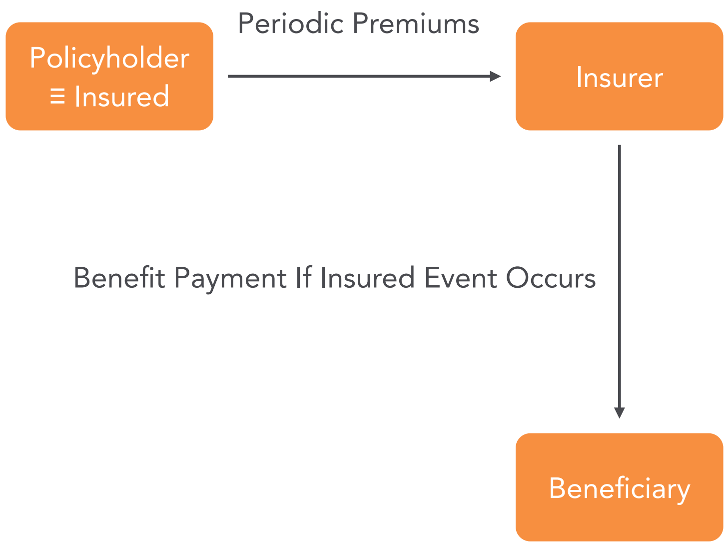



**Figure 1.1:** Illustration of a Life Insurance Contract

The essential features of conventional life insurance contracts are now described. They are the building blocks on which other life insurance contracts are developed.

## 1.2 CONVENTIONAL LIFE INSURANCE

For life insurance contracts, the sum insured (also called benefit payment) is typically declared at the time the policy is issued. Therefore, uncertainty arises from the *timing of death* of the insured. In other words, the future lifetime of the insured is a key random variable for life insurance contracts.

Periodic premium payments made by the insured are also a source of uncertainty for the insurer since the policyholder may face cash flow problems arising from, for example, becoming unemployed or disabled. As a result, the policyholder may be unable to make future premium payments. Obviously, this source of uncertainty does not exist for single premium policies, since full payment is made to the insurer at the time the policy is issued.

To facilitate the presentation for the rest of this chapter, we now present actuarial notation for life insurance and life annuities according to the protocol of the International Association of Actuaries (IAA). This presentation is in Box 1.1 below.

**Box 1.1: Actuarial Notation for Life Insurance (IAA)**

1. **$(x)$ means a life of age x years.** The word 'life' in this context is an actuarial term. We will use 'life' and 'individual' interchangeably in this eBook.

   For example, (40) refers to an individual or life of age 40 years.

2. ***l*** is the number of living; $l_x$ is the number of individuals living who are all of age $x$ years. This is a key statistic in life tables.

   For example, $l_{40}$ is the number of individuals living who are all 40 years of age.

3. **d** is the number of individuals that died; ${}_n d_x \equiv$ number of individuals who died between ages x and x + n.

   For example, ${}_5 d_{40}$ = 1,268 means the number of deaths between ages 40 and 45 years is 1,268.

4. **p** is the probability of living; ${}_n p_x$ is the probability an individual of age **x** years survives **n** years.

   For example, ${}_{10} p_{40}$ represents the probability an individual of age 40 years survives for at least 10 years.

   For the case where n=1, ${}_1 p_x \equiv p_x$. For example, $p_x$ is the probability that a person of age **x** years survives the next year. Therefore, $p_{40} =$ 0.9967 means the probability that a life of age 40 years will attain the age of 41 years is 99.67%.

   $p_x$ is called the *survival rate.*

5. **q** is the probability of dying; $_n q_x$ is probability that an individual currently of age **x** years dies within **n** years.

   For example, $_{20}q_{50}$ represents the probability that an individual of age 50 years dies within the next 20 years.

   For the case where n=1 $_1q_x \equiv q_x$. For example, $q_x$ is the probability that a person of age **x** years will die before the end of the next year.

   $q_x$ is called the *mortality rate*.

6. $\mu_x$ is the probability of an individual aged $x$ years dies the next instant; it is called the **force of mortality at age x.** For example, if $\mu_{60} = 0.0005$, the probability of an individual who has reached the aged of 50 years will die the next instant is 0.05%.

   The force of mortality is called the instantaneous rate of mortality. One may expect that the force of mortality for an individual of age 100 years will be higher for the same individual than when he/she was 60 years of age.

7. The symbol $T_x$ represents the future lifetime of an individual of age $x$ years. $T_x$ is also called ***time-until-death*** for an individual of age $x$ years.

   For example, $T_{40}$ represents the future lifetime of an individual currently 40 years old. Similarly, $T_0$ is the future lifetime of a *newborn*.

   $T_x$ is a random variable meaning that if $T_{40}$ = 50.3 years with probability 30%, there is a 30% chance that an individual of age 40 years will have a future (remaining) lifetime of 50.3 years.

8. Closely related to $T_x$, is the individual's ***age-at-death*** (X). This is the value of the individual's age from the time of birth to the time of death.

   For an individual of age *x* years, the age-at-death (X) is equal to current age (x) plus his/her future remaining lifetime ($T_x$). Therefore:

   $$X = x + T_x$$

   For example, if $T_{40}$ = 50.3 years, then age-at-death is X = 40 years + 50.3 years = 90.3 years.
   This is because the individual has already lived for 40 years and is expected to live another 50.3 years.

9. The **curtate** future lifetime of an individual of age *x* years is denoted by $\mathbf{K_x}$; it represents the number of complete years lived by an individual of $x$ years.

   For example, if the future lifetime of an individual of age *x* years ($\mathbf{T_x}$) is equal to 50.3 years, then $\mathbf{K_x}$ = 50 years which is the *integer* part of 50.3 years. This individual has an expected future lifetime of 50 complete years.

   ***The curtate future lifetime is important for valuing life insurance and life annuities when payment is made at the end of the year of death and thus facilitates the use of life table data.***

**Note:** *We will introduce more actuarial notation when appropriate in the following chapters.*

We now consider two fundamental life insurance contracts from which other life insurance contracts are created. The first is a 'pure endowment insurance' contract that is based on the survival of the individual over a term; that is, for a fixed and finite number of years.

In the insurance literature, it is convention (unless stated otherwise) to assume that the benefit payment (i.e., sum insured) is declared at the date of policy issue and equal to a monetary value of 1.

### 1.2.1 PURE ENDOWMENT INSURANCE

By definition, an *n-year pure endowment insurance* pays a benefit of 1 at the end of the *nth* year if the insured survives *n* years. The next diagram illustrates an n-year pure endowment insurance:


Notation: (x) represents a life of age x years at time 0;
Policy is issued at time 0;
Term of contract is n years.

A pure endowment contract pays a sum insured of 1 at the end of the nth year if the policyholder survives at least n years, ($T_x \geq n$). No payment is made if the insured dies before the term ends.

**Figure 1.2:** Illustration of an n-year Pure Endowment Insurance Contract

From the perspective of the insured, the payoff function for a benefit of monetary value 1 is defined as follows:

$$\mathbf{B} = \begin{cases} \mathbf{1 \text{ if } T_x \geq n} \\ \mathbf{0 \text{ if } T_x < n} \end{cases} \tag{1.1}$$

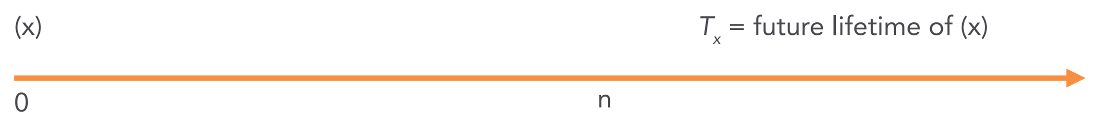
As an illustration, a policyholder of age 40 years purchases a 10-year pure endowment insurance that pays a fixed amount of $50,000; the insurance company pays nothing to the beneficiary if the policyholder dies before 50 years. A payment of $50,000 is paid if the policyholder lives for at least 10 years and thereby reaches an age of 50 years. In actuarial terms, the age-at-death variable $X \geq 50$ years.

**Comment**

A pure endowment is a savings contract where payment of the accumulated value is contingent on the survival of the insured. The actuarial present value of a pure endowment insurance is presented in chapter 4. The other fundamental insurance contract considered is a term life insurance contract.

### 1.2.2 TERM LIFE INSURANCE CONTRACT

An *n-year term life insurance contract* is based on the **death** of the insured. This insurance pays a sum insured or benefit payment equal to 1 if the insured dies within *n* years. This characteristic of term life insurance is illustrated in the next diagram:




Notation: (x) represents a policyholder aged x years;
Policy is issued at time 0;
Term of life insurance policy n years.

A term life insurance policy pays a fixed sum insured if the policyholder dies within n years. (i.e., $T_x<n$). No payment is due if the insured survives the term of the contract.

**Figure 1.3:** Illustration of a Term Life Insurance Contract

From the perspective of the insured, a benefit function (B) is defined as follows:

$$B = \begin{cases} 0 \text{ if } T_x \geq n \\ 1 \text{ if } T_x < n \end{cases} \tag{1.2}$$

Equation (1.2) states that the insured receives a payment that is equal to 1 if he/she dies within n years. If payment is made at the moment of death, it is called the continuous case. If payment is made at the end of the year of death, this is called the discrete (annual) case and the curtate lifetime variable plays a key role in premium calculation principles. These issues are considered in chapter 4.

We consider two other life insurance contracts that are based on the pure endowment and term life insurance described in (1.1) and (1.2).

### 1.2.3 ENDOWMENT INSURANCE

Since the events death and survival are mutually exclusive which means that these events cannot occur simultaneously, then we can take the union of an n-year pure endowment insurance and an n-year term insurance by adding their respective benefit functions stated in (1.1) and (1.2).

The union of these insurance contracts is called an **n-year endowment insurance.** In this case, the benefit function is the sum of the benefit functions in (1.1) and (1.2) resulting in:

$$B = \begin{cases} 1 \text{ if } T_x \geq n; & \text{pure endowment} \\ 1 \text{ if } T_x < n; & \text{term life} \end{cases} \quad (1.3)$$

Equation (1.3) shows that an n-year endowment contract makes a payment of 1 with certainty; that is, there is a payment of 1 if the insured survives at least n years and a payment of 1 if death occurs within n years.

**Comment**


For an endowment insurance, the sum insured for the term life component can be different from the sum insured for the pure endowment component. For example, a policyholder of age 40 years purchases a 10-year endowment insurance pays $50,000 if the insured survives 10 years (the pure endowment component) and pays $100,000 if the insured dies within 10 years (the term life insurance component).

We now consider the case of a life insurance contract without a fixed term so that the insurance expires upon eventual death of the insured. Hence, from the insurer's perspective the occurrence of death of the insured is certain and only the timing of death is random. This is the main characteristic of a **whole life insurance**.

### 1.2.4 WHOLE LIFE INSURANCE

For a whole life insurance, the insurer pays a monetary sum of 1 upon the death of the insured. The main features of a whole life contract are illustrated in the next diagram:

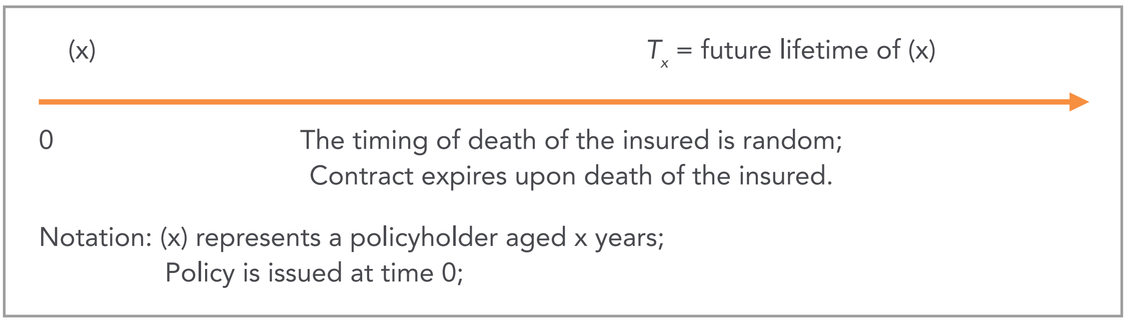



**Figure 1.4:** Illustration of a Whole Life Insurance Contract

We now discuss an insurance contract which may create significant financial risks for the policyholder. This is called *equity-linked life insurance.*

## 1.3 EQUITY-LINKED LIFE INSURANCE

Equity-linked life insurance is called unit-linked insurance in Europe and most of Asia, segregated funds in Canada and variable annuities in the USA.

Equity-linked life insurance (or more accurately called 'investment-linked life insurance') is a contract between a policyholder and an insurance company where the policyholder pays either a single premium or regular premiums until maturity of the contract or upon the death of policyholder, whichever comes first.

The premiums received by the insurance company are invested in a portfolio of assets (e.g., equity, bonds, real estate) or in units of a mutual fund. The insurer makes a payment based on the evolution of the investment over the time period $[0, n]$ where $n$ is the maturity of the equity-linked life insurance contract. If the policyholder is alive at time $n$, then we have the features of a pure endowment insurance. If the policy holder dies before $n$, then we have a term life insurance. This means that an equity-linked insurance is similar to an endowment life insurance.

The payoff for the beneficiary is dependent on the performance of the investment portfolio or mutual fund. Here is a diagrammatic illustration of the benefit options:

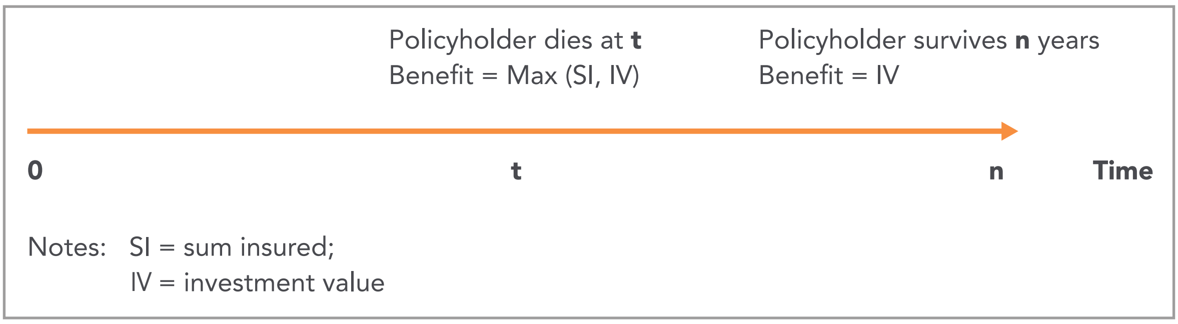



**Figure 1.5:** Benefit Options for Equity Linked Life Insurance

Figure 1.5 shows that if the insured dies at time **t** before the maturity date ($n$), then the benefit payment is typically the higher of two values: the sum insured (SI) declared at the time 0 and the (accumulated) investment value (IV) at time **t**.

If the insured survives the maturity of the contract ($n$), then the benefit payment is typically the accumulated value (IV) of the investment portfolio.

Since the accumulated investment value is not guaranteed, the policyholder bears the financial risk of the investment portfolio fully and solely. This potential for significant financial risk borne by the policyholder can lower the demand for this insurance product and hence have a negative impact on the fee income of the insurer.

For this reason and to create higher demand for this insurance product, the insurer may provide guarantees and options to the policyholder that serve to transfer part or all of the financial risk from the policyholder to the insurance company. This is a noteworthy point that bears emphasis.

**Important Point**

**While insurers typically bear insurance risk such as mortality risk when offering conventional insurance contracts, equity-linked insurance may confer significant financial risk to the policyholders. The provision of guarantees and options to the policyholder transfers part or all of the financial risk to the insurer.**

The next section describes the potential exposure of the insurer to significant financial risk when guarantees and options are added as riders to equity-linked insurance.

### 1.3.1 GUARANTEES AND OPTIONS

Guarantees and options attached as riders to equity-linked insurance are commonly referred to as *GMxBs* meaning *guaranteed minimum benefits of type x.* For example, GMDB refers to guaranteed minimum death benefit and GMWB is an acronym for guaranteed minimum withdrawal benefit. These are two examples of numerous possibilities of guarantees and options.

These guarantees are riders associated with equity-linked insurance contracts and come under a common name: *variable annuities (VA).* For example, SPVA with a rider GMWB means a single premium variable annuity with a guaranteed minimum withdrawal benefit. This is essentially an equity-linked insurance product with a single premium and provides a guaranteed minimum withdrawal benefit.

Box 1.2 illustrates the mechanics of a SPVA with GMWB and highlights the exposure of the insurance company to financial risk.

**Box 1.2: Example of SPVA with GMWB**

A policyholder purchases €100,000 SPVA with GMWB with a term of 20 years and a withdrawal benefit of 5% of the initial premium (or €5,000) regardless of the investment fund performance.

This means that in case the account value reaches a value of zero before the end of the term of the SPVA, the insurance company is obligated to pay the policyholder withdrawal benefits of €5,000 each year for the remaining term.

For purposes of this example, we assume that the policyholder survives the term of the SPVA.

The table below reflects the investment performance of the fund based on an initial premium of €100,000 and where the equity markets experience a four-year period of extreme losses. Explanations follow this table.

| Contract Year | Investment Return | Accumulated Fund Value | Withdrawal | Accumulated Fund Value after Withdrawal |
|---|---|---|---|---|
| 1 | 10% | 1.10 x €100,000 = €110,000 | €5,000 | €105,000 |
| 2 | 10% | 1.10 x €105,000 = €115,500 | €5,000 | €110,500 |
| 3 | -25% | 0.75 x €110,500 = €82,875 | €5,000 | €77,875 |
| 4 | -40% | 0.60 x €77,875 = €46,725 | €5,000 | €41,725 |
| 5 | -60% | 0.40 x €41,725 = €16, 690 | €5,000 | €11,690 |
| 6 | -50% | 0.50 x €11,690 = €5,845 | €5,000 | €845 |
| 7 | +20% | 1.20 x €845 = €1,014 | €5,000 | - €3,886 |

**Comment**

a) For contract year 1, the initial premium of €100,000 is invested in a fund of risky assets that earned 10% at the end of the year. The ending fund value before withdrawal benefit is €110,000. The policyholder withdraws €5,000 and the accumulated fund value at the end of the year after the withdrawal is €105,000.

   This value of €105,000 is the starting investment for the second year which earned 10% and the explanation for the calculations for year 2 is similar to those for year 1 presented in the paragraph above. The accumulated value at the end of period 2 after withdrawal is €110,500.

   Dramatic declines in the equity market take place from year 3 through year 6.

b) At the end the 7$^{th}$ year, the account value is negative and the insurer is obligated to subsidise the withdrawal for this year to the amount of €3,886. The insurer must also pay the policyholder €5,000 for each for all subsequent years until the end of the 20-year term.

c) The GMWB has given the policyholder the following key benefit:
   *There is downside protection for the policyholder against financial risk while maintaining upside gains of the investment portfolio.*

***The insurer has absorbed risk of providing downside protection for the policyholder. This has the potential for significant financial losses for the insurer.***

We close this chapter with a discussion of life annuities. To facilitate the discussion, we provide a brief review of certain annuities where the term and level periodic payments are known and fixed at the contract date.

A complete presentation of annuities is found in the first eBook in the ART of INSURANCE series entitled *Principles of Insurance.*

## 1.4 ANNUITIES CERTAIN

Formally, a **certain** annuity is a contract between an individual and a financial institution (e.g., bank) where the individual receives payments from the financial institution that are fixed over equally-spaced intervals (e.g., months) over a known finite period of time. This annuity is called **annuity certain** in actuarial terminology. In exchange, the institution receives a single premium from the individual at the date the contract is issued. The individual is called an **annuitant**.

Simply put, a certain annuity is a stream of payments paid to the annuitant for a finite term. The present value of the payments made by the financial institution is the single premium paid by the individual.

Box 1.3 reviews the calculation of the present value of payments made at the end of the period (called **annuity immediate**) and at the beginning of the period (called **annuity due**) as well as **deferred annuities**. At this time, we present additional actuarial notation completing Box 1.1 above.

**Box 1.3: Actuarial Notation for Annuities (IAA)**

We begin with an annuity immediate, and consider the calculation of both the present and future value for a constant interest rate *(i )*. The constant interest rate is also called the *technical interest rate.*

**Present Value of an Annuity Immediate**

A financial transaction promises to pay a constant and equal amount of money at equally-spaced time intervals (e.g., years) for a specified number of time periods (e.g., 10 years).

The following diagram illustrates a financial transaction involving a payment of 1 each year for *n* years where payment is made *at the end of each year*. This stream of payments is called an **annuity immediate**. The periodic payment and the number of such payments are both certain.

Figure 1.6 presents a graphical depiction of these periodic cash flows each of 1.

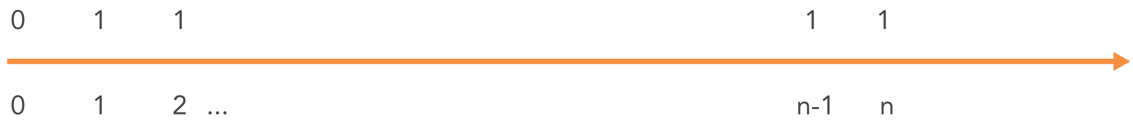



**Figure 1.6:** Annuity Immediate

In actuarial notation, the present value of an annuity immediate shown in equation (1.4) below is represented by $\mathbf{a}_{\overline{n}|i}$. For this symbol, **a** represents present value of an annuity immediate, **n** is the number of annual payments and ***(i)*** is the effective annual interest rate.

The present value of an annuity immediate for equal periodic payments of 1 and interest rate *(i)* is given by the formula:

$$\mathbf{a}_{\overline{n}|i} = \frac{1 - v^n}{i} \text{ where } v \equiv \frac{1}{1+i}. \qquad (1.4)$$

Here is an example which illustrates this equation.

**Example 1**
An annuity immediate pays an annual amount of $1 for a period of 10 years. The interest rate is 6%. The present value of the annuity immediate is:

$$a_{\overline{10}|0.06} = \frac{1-\left(\frac{1}{1+0.06}\right)^{10}}{0.06} = \$7.360087$$

We now consider the case of an annuity where the equal periodic payments are made at the **beginning** of each period. This is called an **annuity due**. These annuities have important applications to the valuation of life annuities in chapter 3 and life insurance in chapter 4.

**Present Value of an Annuity Due**
Here is a graphical depiction of an annuity due for *n* periods and equal periodic payments of 1.

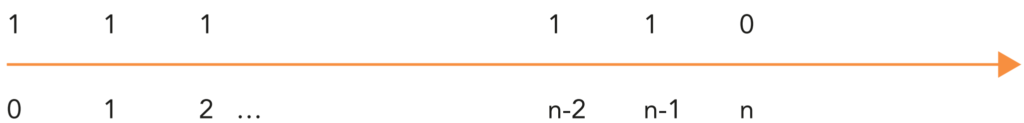



**Figure 1.7:** Annuity Due

Observe from Figure 1.7 that payments begin at time 0 and end at time *n-1.*

In actuarial notation, the present value of an annuity due shown in equation (1.5) below is represented by $\ddot{a}_{\overline{n}|i}$. In this case the symbol **ä** refers to present value of an annuity due where the first payment is made immediately. The two dots placed over a letter is called a *trema.*

The present value of an annuity due is given by the formula:

$$\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d} \text{ where } v \equiv \frac{1}{1+i} \text{ and } d = iv \tag{1.5}$$

Here is an example which illustrates the application of this equation.

**Example 2**
An annuity due comprises an equal annual payment of $1 at the beginning of each year for 10 years. The interest rate is 6%. The present value factor is $\frac{1-\left(\frac{1}{1+0.06}\right)^{10}}{0.06\times\frac{1}{1+.06}} = \$7.801692$.

**Intuition**
Note that from examples 1 and 2, the present value of an annuity due is higher than the present value for the corresponding annuity immediate. This is intuitive since an annuity due makes the first payment immediately while payments start at the end of year for an annuity immediate.

(Money received earlier has a higher (present) value that money received later in the future).

Using equations (1.4) and (1.5), it is shown that:

$$a_{\overline{n}|i} = v \times \ddot{a}_{\overline{n}|i}\ . \tag{1.6}$$

**Example 3**

The present value of a 10-year annuity due is 7.8017 and interest rate is 6% so that v = 0.9434. What is the implied present value of a 10-year annuity immediate for the same interest rate?

The answer is found from (1.6) where the present value of a 10-year annuity due is 7.8017 and v = 0.9434. Hence $a_{\overline{n}|i}$ = 0.9434 x 7.8017 = 7.3600

We close this review by discussing **deferred annuities**.

**Deferred Annuities**

For a deferred annuity, the first payment to the annuitant begins at a point in the future. For example, a *3- year deferred annuity* means that the first payment begins 3 years from today.

Since a regular annuity due makes the first payment at time 0, then a 3-year deferred annuity due will make the first payment at the beginning of third year from today.

We provide an illustration and calculation procedure for a k-period deferred annuity due.

**Present Value of a k-period Deferred Annuity Due with a Term of n Periods**

The calculation of the present value of a k-period deferred n-year annuity due involves two steps.

The first step is to consider the present value of a regular annuity due over n-periods starting k periods from today. The time period between k and k + n is shown by the red arrow below (Figure 1.8).


**Figure 1.8:** Deferred Annuity due

The second step is to discount the value of the annuity due for **k** periods (see grey arrow). This means that we multiply the value of the annuity due by a discount factor for **k**. This discount factor is, $v^k = \left(\frac{1}{1+i}\right)^k$.

This gives the present value of a **k**-period deferred annuity due as follows:

$$v^k \ddot{a}_{\overline{n}|i} = v^k \times \frac{1-v^n}{d} \qquad (1.7)$$

The actuarial notation for the present value of a k-period deferred annuity due is ${}_{k|}\ddot{a}_{\overline{n}|i}$. In this symbol, the letter **k** followed by a perpendicular bar indicates that the annuity is deferred for **k** periods in the future – that is, the first payment begins at the start of the $k^{th}$ period. The remaining part of the symbol is explained already. It symbolises an annuity due for **n** periods and effective interest rate ***(i)***.

Hence we can state that: $${}_{k|}\ddot{a}_{\overline{n}|i} = v^k \times \frac{1-v^n}{d} \qquad (1.8)$$

Here is an example to explain the implication of formula (1.8).

**Example 4**

Refer to the information provided in Example 2 where $1 paid annually, n=10 years and interest rate = 6%. It is shown that the present value of this annuity due is $7.802692

What is the present value of this annuity if the first payment is deferred for 3 years?

Here is a diagram of this 3-year deferred annuity due:


From (1.8), PV (3-year deferred annuity due) $= v^3 \times \$7.801692 = \left(\frac{1}{1.06}\right)^3 \times \$7.801692 = \$6.550451$

**Intuition**

Observe that the present value of a deferred annuity due is $6.550451 which is less than the present value corresponding annuity due that is not deferred. This value is $7.801692. This is intuitive since an annuity that makes payments immediately is more valuable than a similar annuity that makes the first payment sometime in the future (i.e., deferred).

We now consider life annuities for the **case of a single life** and **payments calculated on the future lifetime of the annuitant**. As shown in Box 1.3 above, annuities certain have payments by the financial institution that are made for a fixed and known time period. This is different for life annuities where payments are made by an insurance company that are based on the future lifetime of the **annuitant** – that is, as long as the annuitant survives.

## 1.5 LIFE ANNUITIES

Before introducing commonly-used actuarial notation for life annuities, the next diagram illustrates the generic process underlying life annuities:


**Figure 1.9** Mechanics of a Generic Life Annuity

### 1.5.1 TERM LIFE AND WHOLE LIFE ANNUITY DUE

A life annuity is a contract wherein an individual of age *x* years receives periodic payments from an insurance company in exchange for a single premium paid to the insurance company at the date of contract issue. The periodic payments are typically made on a monthly basis at the beginning of the month for as long as the annuitant survives.

**If payments are made at the beginning of the period for as long as the individual is alive, the life annuity is called a whole life annuity due.**

**If the life annuity contract is for a fixed term of *n* years, the annuitant receives payments at the beginning of each period for a term no longer than *n* years. Payments cease before *n* years, if the annuitant dies before the term of the contract. This is called a temporary life annuity due.**

Here is an example that illustrates the main attributes of life annuities.

Consider the case of a whole life annuity which makes a payment at the beginning of each month for as long as the annuitant survives. An individual of age 65 years has reached retirement with an accumulated sum of €500,000. The individual elected to annuitise his/her accumulated value with an insurance company which is contracted to pay 0.5% of the initial investment of €500,000 at the beginning of each month as long as the annuitant survives. This is an example of a whole-life annuity due with a guaranteed monthly payment of €2,500.

There are several implications from this simple example which, for the time being, is based on mortality only. We discuss these implications next:

a) For a whole life annuity, the annuitant pays a fixed premium at the beginning of the contract and receives periodic payments until the annuitant dies. **The risk to the insurance company is that insured lives longer than expected.** In this case, the insurer makes more total payments than expected.
It is worth noting that by comparison, for a whole life insurance policy, the policyholder makes periodic payments until the insured dies. The insurance company makes a final payment that is equal to the sum insured to the beneficiary. **The risk to the insurance company is that the insured dies earlier than expected.** Hence, the insurer receives less total premiums than expected.

b) As indicated in a) above, there is risk to the insurer that an annuitant lives longer than expected.
However, the **principle of mutuality** serves to mitigate this risk to the insurer since those annuitants that die earlier than expected subsidise those annuitants who live longer than expected.

c) Annuitants take the risk that they will die early into the contract and lose all future payments from the insurer. Refer to our example above where €500,000 was annuitised into €2,500 monthly payments. If the annuitant dies after even

one month, then all future payments to the beneficiary are lost. This possibility can have a negative effect on the demand for life annuities.
Not surprisingly, insurers typically attach riders to life annuities that serve to mitigate this problem for potential annuitants. Here are two examples.

### 1.5.2 LIFE ANNUITY WITH PERIOD CERTAIN.

For illustration, assume a whole life annuity with a 10-year **period certain**. Since it is a whole life annuity, payments to the beneficiary will be made until the death of the annuitant. But if the annuitant dies at the end of year 1, then payments will be made to the designated beneficiary for the next 9 years. The 10-year period certain is a guaranteed period of payments and so reduces the risk of the annuitant dying early and losing a significant amount of future payments.

**Comment on Period Certain**

Payments are made for a guaranteed period even if the annuitant dies before the end of this period.

Clearly, the longer the guaranteed period, the lower the amount of periodic payments to the beneficiary.

One other way of extending payments (typically at a subsequent reduced level) is through a last survivor life annuity.

### 1.5.3 JOINT LIFE AND LAST SURVIVOR LIFE ANNUITIES

For simplicity, we consider the case of two-lives (e.g., spouses).

For a joint life annuity, payments are made until one of the two spouses dies. That is, payments are made as long as **both** spouses are alive and ceases upon the **first** death.

For a last survivor annuity, payments continue until the **last** death of the spouses. That is, payments continue as long as one spouse is alive. Typically, the payment made when both spouses are alive is reduced after one of the two spouses dies. All payments cease after the death of the second spouse.

This chapter is concluded. Chapter 2 discusses approaches to estimating mortality probabilities required for valuing life annuities (Chapter 3) and life insurance (Chapter 4).

# 2 MORTALITY MODELS AND LIFE TABLES

## 2.1 INTRODUCTION

It is noted in chapter 1 that for insurance contracts, benefit payments are made either at the moment of death or at the end of the year of death. Life insurance contracts where payments made at the moment of death (i.e., the continuous case) involve advanced calculus methods and probability theory. The second case where payments are made at the end of the year requires relatively simple algebraic methods and importantly, permits the use of life table data. This significantly simplifies the calculation methods and facilitates intuition and greater insight of insurance principles and concepts. Accordingly, and in keeping with our preferred approach, we assume that payments are made at the end of the year of death. We consider the continuous case in subsequent eBooks in Level II of the Art of Insurance series.

In simple terms, a life table tracks the evolution of mortality rates over time for a population. Life tables may be created for a group of persons with one of more common characteristics (e.g., relatively close birthdates) called a cohort. A cohort life table provides actual age-specific mortality rates over the lifetime of a specific cohort. Simply put, a cohort life table is based on actual mortality experience of the cohort from birth until the cohort has no remaining living members. It is apparent that a cohort life table will take a long time to complete – over 100 years in developed countries and so remains a drawback.

A period life table predicts the future mortality experience over the lifetime of a cohort based on the actual mortality experience during a recent period of time. For example, a period life cycle for 2010 may be based on the average mortality experience of 2008, 2009 and 2010 and future mortality rates are predicted based on **this** average mortality experience. Since mortality rates are dynamic and can experience significant systematic changes over time, period tables may lead to high prediction errors. For this reason, period life tables are typically updated to reflect recent systematic changes in mortality rates.

We begin this chapter by explaining the information content of a typical life table using the **Illustrative Life Table** (hereafter, **ILT**) as the source for examples illustrating key life table statistics. The ILT is usually provided by the Exam Committee of the Society of Actuaries for candidates taking the accredited examination on ‘Models for Life Contingencies. In this eBook, we consider data from the Spring 2017 version.

Section 2.2 focuses mainly on the classical De Moivre law of mortality and its resulting survival probability function. This probability function links the number of survivors between ages and hence facilitates the prediction of future mortality rates. We selected this law of mortality because of its computational ease and because it is also widely used in the actuarial literature. The De Moivre model provides a simple survival function which is central in the creation of a life table. It also provides simple calculation procedures for important statistics such as expected lifetime, force of mortality and premium calculation of life insurance and life annuities.

## 2.2 INFORMATION CONTENT OF A LIFE TABLE

Before we discuss information content of ILT, we introduce additional actuarial notation related to life tables in Box 2.1.

**Box 2.1 Actuarial Notation for Life Tables (IAA)**

1. $l_0$ = the number of newborns (e.g., 100,000) in the cohort; this is called the **radix** of the life table. As stated in Box 1.1, the letter $l$ represents the number of living and so $l_{40}$ is the *expected* number of individuals who attained the age of 40 years.
2. The maximum lifetime for any life is represented by the symbol, $\omega$, called the *ultimate age*. The number of lives at age $\omega$ is zero; that is, $l_\omega = 0$. The ILT assumes $\omega$ = 120 years.
3. The number of lives remaining up to age *x* years is represented by the symbol, $l_x$. Obviously, the number of lives declines over time with a maximum value at the radix and a zero value at the ultimate age, $\omega$.

   The decline in $l_x$ over time is due to some deaths in the cohort. At later ages, the mortality rate increases with higher rates of increase at later ages.

For reference, we introduce an extract of the ILT which will be used to highlight, for example, the calculation procedures of survival and mortality probabilities.

| Age (x) | Number of living at Age x; $l_x$ | Number of Deaths, $d_x$ |
|---|---|---|
| 0 | 100,000 | 2,042 |
| 1 | 97,958 | 132 |
| 2 | 97,826 | 120 |
| … | … | … |

| | | |
|---|---|---|
| 20 | 96,178 | 99 |
| 21 | 96,079 | |
| … | | |
| 40 | 93,132 | 259 |
| 41 | 92,873 | |
| … | | |
| 50 | 89,509 | 530 |
| 51 | 88,979 | |
| … | | |
| 60 | 81,881 | |
| … | | |
| 65 | 75,340 | |
| … | | |
| 100 | 400 | 163 |
| 101 | 237 | |

**Table 2.1:** An Excerpt of the Illustrative Life Table

We now discuss the information content of the ILT model presented in Table 2.1

1. The illustrative life table shows that number of newborns ($l_0$) is equal to 100,000 which is the radix of the life table. This is number of newborns at the beginning of the first year.
2. The number of individuals living at the beginning of the second year (i.e., one year from birth) is $l_1$=97,958 relative to the 100,000 at the beginning of year 1. Therefore, the number of deaths during the first year is 100,000 – 97,958 = 2,042. Here is a timeline illustrating this information:

Formally, the number of deaths between ages $x$ and $x+1$ is represented in actuarial notation as follows: ${}_1d_x = l_x - l_{x+1}$. Based on this equation, $l_0$ = 100,000 and $l_1$ = 97,958 and so the number deaths between ages 0 and 1 is ${}_1d_0 = l_0 - l_1 =$ 2,042.
(Recall from chapter 1, ${}_1d_x \equiv d_x$)

3. More generally, the number of deaths between ages x and x + t is given as follows:

$$ {}_td_x = l_x - l_{x+t} \tag{2.1} $$

For example, using data from the ILT model in Table 2.1, the number of deaths between years 20 and 50 is given by ${}_{30}d_{20} = l_{20} - l_{50} =$ 96,178 – 89,509 = 6,669.

4. The probability that an individual of age $x$ years dies within the next $t$ years is:

$$ {}_tq_x = \frac{{}_td_x}{l_x} \tag{2.2} $$

Equation (2.2) is intuitive. It states that the probability of dying within the next $t$ years is calculated as the number of deaths over the period of $t$ years expressed as a fraction of the number of individuals living at the start of the period.

Substituting equation (2.1) into (2.2), we obtain:

$$ {}_tq_x = \frac{l_x - l_{x+t}}{l_x} \tag{2.3} $$

**Example 2**

Refer to the information in the ILT in Table 2.1. Calculate the probability that a life aged 20 years dies within the next 30 years.

By (2.3), we calculate ${}_{30}q_{20} = \frac{l_{20} - l_{50}}{l_{20}} = \frac{96{,}178 - 89509}{96{,}178} = \frac{6{,}669}{96{,}178} =$ 0.06934 = 6.934%.

5. From the mortality probability shown in equation (2.3), the survival probability can be inferred. The survival probability refers to an individual of age $x$ years who lives at least another $t$ years.

Since an individual will either die or survive in a finite time period, then the corresponding probabilities must add to 100%. In other words, the survival probability (${}_tp_x$) and the corresponding mortality probability (${}_tq_x$) add to 100%. Hence, for an individual who is of age $x$ years, the probability that he/she will survive at least $t$ years is:

$$ {}_tp_x = 1 - {}_tq_x \tag{2.4} $$

Equivalently, substituting equation (2.3) into equation (2.4) results in (2.5) below:

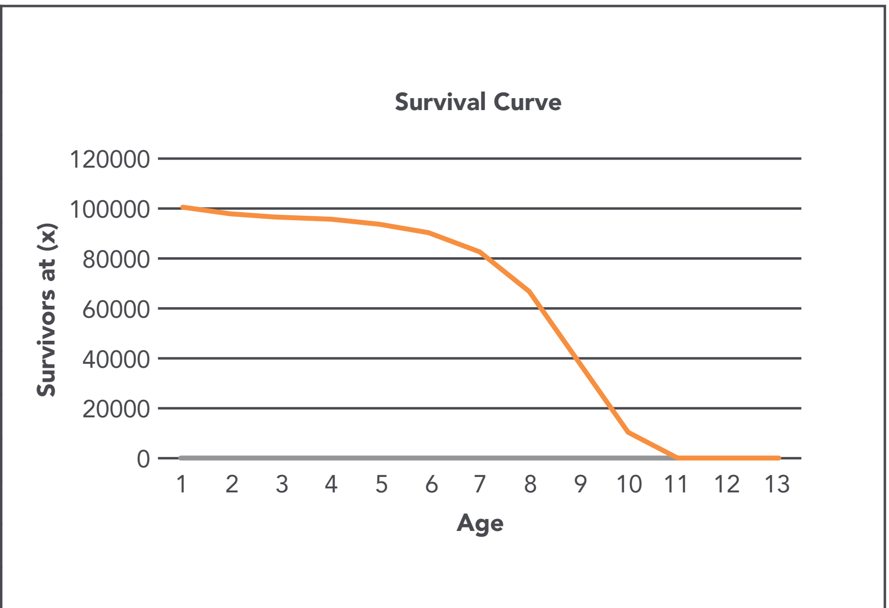
$$ {}_t p_x = \frac{l_{x+t}}{l_x} \qquad (2.5)$$

**Example 3**

By (2.4), the probability that for an individual of age 20 years lives for at least an additional 30 years (i.e., ${}_{30}p_{20}$) is 100% – 6.934% = 93.066%. Alternatively, using equation (2.5): ${}_{30}p_{20} = \frac{l_{50}}{l_{20}} = \frac{89{,}509}{96{,}578} = 93.066\%.$

It is insightful to observe the survival rates over time from the ILT. Here is a graph of $l_x$ for the illustrative life table for selected values of ages ($x$) for 10-year intervals.

| Age (x) | Living to x |
|---|---|
| 0 | 100000 |
| 10 | 97055 |
| 20 | 96178 |
| 30 | 95013 |
| 40 | 93132 |
| 50 | 89509 |
| 60 | 81881 |
| 70 | 66161 |
| 80 | 39144 |
| 90 | 10585 |
| 100 | 400 |
| 110 | 11 |
| 120 | 0 |




**Graph 2.2** A Survival Graph Based on the Illustrative Life Table

**Comment**

Graph 2.2 shows that the slope of the survival curve is negative indicating a decline in the number of survivors out of the initial 100,000 newborns over time. This is expected since there are deaths over time. But the graph is relatively flat (i.e., with almost zero slope) with a sudden sharp decline in the number of survivors after about age 70. Equivalently, mortality rates are relatively low up to about age 70 years and then experience a sharp increase.

We close this section by considering another example that reviews all concepts introduced so far in this chapter.

**Example 4**

This example is based on an excerpt from the (period) life table for the US Social Security area male population (2014).

| Exact Age (x) | $l_x$ | $q_x$ |
|---|---|---|
| 0 | 100,000 | 0.006322 |
| 1 | 99,368 | 0.000396 |
| 20 | 98,802 | ? |
| 21 | 98,701 | 0.001151 |
| 50 | 92,913 | 0.004987 |
| 100 | 1,001 | 0.349027 |

From this information, we provide answers for selected questions.

a) What is the value of $d_0$?

From equation (2.1), $d_0 = l_1 - l_0 =$ 100,000 – 99,368 = 632

b) What is the probability that an individual of age 20 years dies over the next year ($q_{20}$)?

From equation (2.3), $q_{20} = \frac{l_{20} - l_{21}}{l_{20}} = \frac{98802 - 98701}{98802} = \frac{101}{98802} = 0.001022$

c) What is the probability that a newborn will survive one year?

By (2.5) $p_0 = \frac{l_1}{l_0} = \frac{99368}{100{,}000} = 0.99368$

d) What is the probability that an individual of age 20 years will survive another 80 years?

From equation (2.5), ${}_{80}p_{20} = \frac{l_{100}}{l_{20}} = \frac{1{,}001}{98{,}802} = 0.01013$

An important concept that is required for determining appropriate premiums for life insurance (considered in chapter 4) is called **deferred mortality**.

To illustrate this concept, consider an individual who is of age 20 years. We want to estimate the probability that he/she lives for the next 50 years and then die within the next 5 years? Note that this is an example where mortality is deferred to a period **after** a period of survival.

More generally, we seek an answer to the question: *for an individual of age x years what is the probability that he/she survives t years and dies within the next u years?*

### 2.2.1 DEFERRED MORTALITY

Here is a graphical depiction of deferred mortality for a life (x) that survives t years and dies within u years.


Notes:

a) Probability that an individual of age x years survives t years is $_tp_x$. This refers to the first part of the time line coloured in red.

b) After t years, the life has reached an age of x + t years. See grey-coloured time line. Hence, the probability that the individual aged x+t dies within u years is $_uq_{x+t}$.

**Figure 2.1:** Deferred Mortality

Using the actuarial notation in Box 1.1, the probability that an individual of age *x* years survives *t* years and dies within the next *u* years is symbolised as $_{t|u}q_x$. From Figure 2.1, it is observed that the deferred mortality is the product of the survival probability and the mortality probability as follows:

$$_{t|u}q_x = {}_tp_x \times {}_uq_{x+t} \tag{2.6}$$

Note that the individual of age *x* years lives for another *t* years and hence reaches *x* + *t* years. Equation (2.6) obtains the probability that the life (x + t) dies within *u* years.

Using life table data and equations (2.3) and (2.5), we obtain:

$$_tp_x = \frac{l_{x+t}}{l_x}; \quad {}_uq_{x+t} = \frac{l_{x+t} - l_{x+t+u}}{l_{x+t}}$$

If we multiply the life table expression for these two probabilities, we obtain the deferred probability as follows:

$$_{t|u}q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x} \tag{2.7}$$

*Equation (2.7) is central for the valuation of life insurance contracts and it requires only life table data.*

**Example 5**

For an individual of age 40 years, what is the probability that he /she dies within 10 years after surviving the next year. Hence *x =50, t=1 years and u=10 years.*

From the ILT in Table 2.1, we obtain, $l_{40}$ =93,132; $l_{41}$ = 92,873 and $l_{51}$ = 88,979. Hence from (2.7), ${}_{10|1}q_{40} = \frac{l_{41} - l_{51}}{l_{40}} = \frac{92{,}873 - 88{,}979}{93{,}132} = 0.041812$.

This completes our introduction dealing with the information content of ILT.

We now consider link between survival functions and life tables.

In the above discussion, it is observed that a life table shows the evolution of the number of survivors over time from the initial number of newborns.

But how is this evolution determined? In other words, what determines the transition of survivors from one year to another?

In the ILT in Table 2.1 the number of newborns is 100,000 and the number of newborns living one year later is 97,958 inferring that 2,042 deaths during the first year.

How is this information obtained?

What is the link between the number of survivors in one year and the number of survivors one year later?

The answers to these questions depend on a mortality model and its resulting survival function.

We explain the key insights as follows:

A life table model links the expected number of individuals of age $t$ years, $l_t$ out of the number of newborns, $l_0$ through a survival function, $S_0(t)$. Here is an illustration of this process:

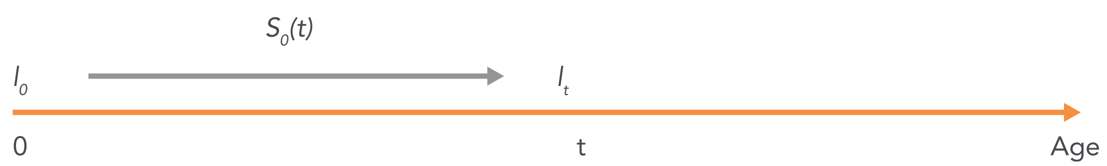



**Figure 2.2:** The Transition Process via a Survival Function

Figure 2.2 is a diagrammatic representation of equation (2.8) as follows:

$$l_t = l_0 \times S_0(t) \qquad (2.8)$$

where $S_0(t)$ is a survival function.

How is this survival function derived?

A survival function is derived from a model of mortality. The classical De Moivre model of mortality presented in the next section of this chapter, describes a specific survival function. Other mortality models have associated survival functions that can be quite complex. We choose the De Moivre model for its simplicity while permitting a full discussion of the main issues in life insurance without having to resort to advanced mathematics.

### 2.2.2 DE MOIVRE LAW OF MORTALITY

A seminal mortality model due to Abraham de Moivre was first published in his 1725 textbook, *Annuities upon Lives*. The key assumption for this model is:

***The random variable, age-at-death X is uniformly distributed over the future life of the newborn. This is also summarised as UDD meaning uniform distribution of deaths.***

Before we discuss this model any further, we review the main characteristic of a uniform distribution.

Figure 2.3 shows the uniform distribution of X as a rectangular shape where each age on the X- axis ranges between X= 0 (i.e., newborn) and X = ω (i.e., ultimate age or maximum lifetime). For the vertical axis, the probability that X is equal to *any* value in this range [0, ω] is equal to $\frac{1}{\omega}$. This is because the area of the rectangle represents total probability of 100%. Since the width of the rectangle is ω, the height of the rectangle must be $\frac{1}{\omega}$ so that the area which is product of width and height is equal to 1.

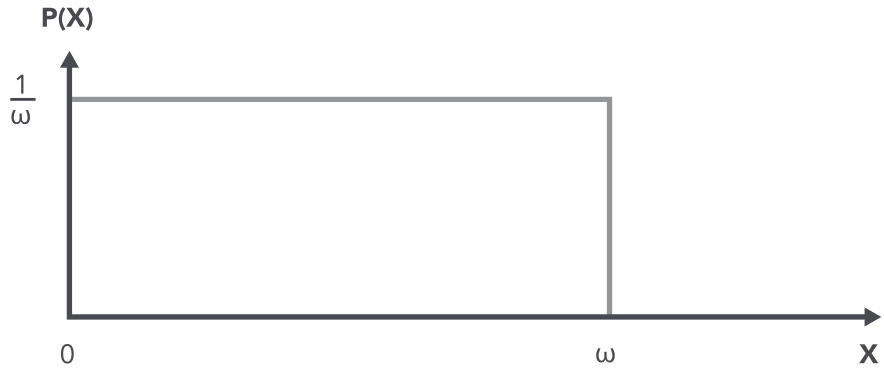



**Figure 2.3:** Uniform Distribution of the Age-at-Death Variable (X)

We now introduce the concept of a survival function which describes evolution of the future life of a **newborn.**

To obtain the survival function for a newborn, we want to find the probability that a newborn will survive **t** years; equivalently, the probability that the age-at-death (X) is higher that *t* years. Formally, a survival function, $S_0(t)$ is stated as follows:

$$\mathbf{S_0(t) = P(X > t)} \tag{2.9}$$

We now intuitively evaluate equation (2.9) for the De Moivre model where X is assumed to be uniformly distributed over the range [0, ω] and represented in Figure 2.4 below.

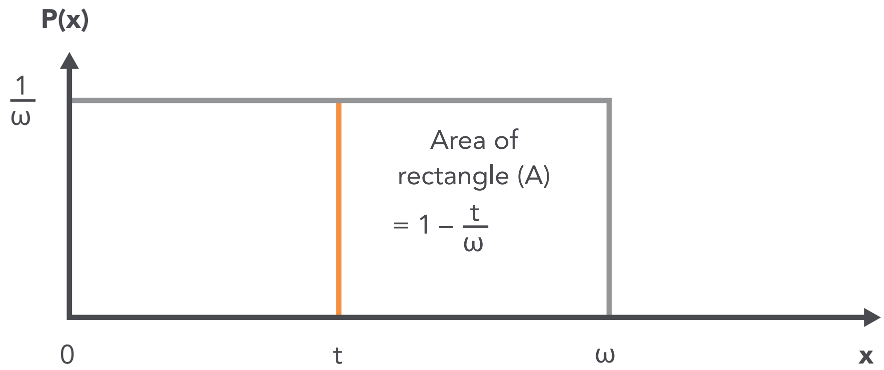



**Figure 2.4:** Survival Function for the De Moivre Model

The survival function $S_0(t)$ is represented by the area of the rectangle to the **right** of the orange-coloured vertical line which corresponds to **X = t**. This is because we want the probability that X > t. This rectangle is labeled A in Figure 2.4. The area of rectangle A is width × height = $\mathbf{(\omega - t)} \times \frac{1}{\boldsymbol{\omega}} = 1 - \frac{\mathbf{t}}{\boldsymbol{\omega}}$.

This is a key result which is summarised as follows:

The probability that a newborn will survive **t** years is given by the survival function:

$$\mathbf{S_0(t)} = 1 - \frac{\mathbf{t}}{\boldsymbol{\omega}} \tag{2.10}$$

**Example 6**

For an ultimate age of ω =120 years, the probability that a newborn will survive 70 years is given by equation (2.10) as follows: $S_0(70) = 1 - \frac{70}{120}$ = 0.4167 = 41.67%. This is much lower than in the ILT in graph 2.2 mainly because the ILT is based on a different mortality model and hence, a different survival function.

**Comment**

The result in example 6 means that out the initial number of newborns of 100,000, the number of survivors at age 70 years is 41.67% of 100,000 = 41,670. This result is also obtained from (2.8) above.

**Comment**

The formula for the survival function in (2.10) is a probability statement used by statisticians. But actuaries use a different symbol for the same expression. Recall from Box 1.1 of Chapter 1, $_t\mathbf{p}_0$ is the probability that a newborn will survive at least **t** years. But this is precisely the survival function, $\mathbf{S}_0\mathbf{(t)}$ in (2.10).

Hence, bringing statistical and actuarial notation together we can state that:

$$_t p_0 \equiv S_0(t) \tag{2.11}$$

Using (2.10), we obtain:

$$_t p_0 \equiv \left(1 - \frac{t}{\omega}\right) \tag{2.12}$$

We have now answered the question posed above for the De Moivre model of mortality: what determines the transition of survivors from one period to another?

$$\text{The answer is } l_t = l_0 \times {}_t p_0 \text{ where } {}_t p_0 \equiv \left(1 - \frac{t}{\omega}\right) \tag{2.13}$$

**Example 7**

How many newborns survived 70 years if the radix of the life table is $l_0$ = 100,000? From equation (2.13), $l_{70} = l_0 \times {}_t p_0 = 100,000 \times (1 - \frac{70}{120}) = 100,000 \times \frac{50}{120} =$ 41,667.

To this point, we considered the survival function for newborns. However, insurance policies are sold to individuals who are of legal age. Hence it is necessary to generalise the results for newborns in this section for individuals of age *x* years, that is for (x).

### 2.2.3 SURVIVAL FUNCTION FOR (X): LIFE OF *X* YEARS

We want to calculate the probability that a person of age $x$ years will survive at least $t$ years or attain the age of **x + t** years. Here is a diagram (Figure 2.5) showing the relevant range for the uniform distribution of X is [x, ω]. The width of the rectangle is **(ω-x)** which means that the height must be $\frac{1}{\omega - x}$ ensuring that the area is equal to 1.

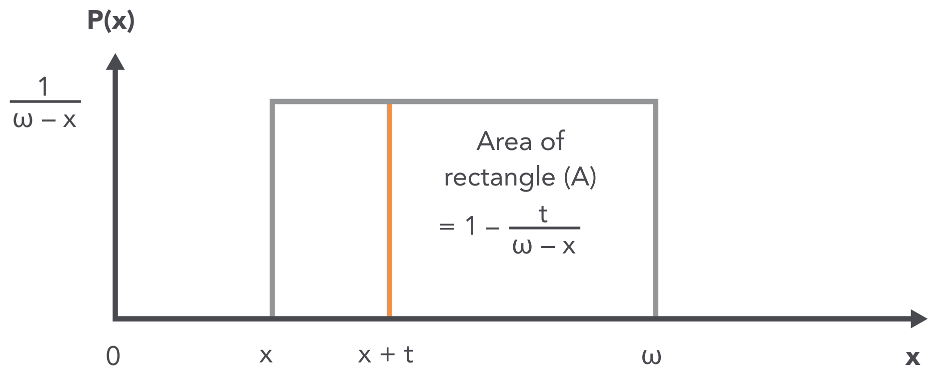



**Figure 2.5:** Survival Function for the De Moivre Model

Our discussion is similar to that in the previous section. The survival function $\mathbf{S_x(t)}$ is the area of the rectangle in Figure 2.5 that is to the right of the orange coloured-vertical line at age **x + t.** This is the area of rectangle (A) which is equal to width × height = $\mathbf{(\omega - (x + t))}$ × $\frac{1}{\omega - x} = \frac{\mathbf{\omega - (x + t)}}{\mathbf{\omega - x}} = 1 - \frac{\mathbf{t}}{\mathbf{\omega - x}}$.

**Comment**

For an individual of age **x,** the survival function **t** years later (i.e., at age **x + t)** is given by equation (2.14) below:

$$\mathbf{S_x(t)} = 1 - \frac{\mathbf{t}}{\mathbf{\omega - x}} \tag{2.14}$$

Hence, the evolution of survival rates of individuals in the cohort is given by the equation:

$$l_t = l_x \times S_x(t) \tag{2.15}$$

Substituting (2.1.4) into (2.1.5), we obtain:

$$l_t = l_x \times \left(1 - \frac{t}{\omega - x}\right) \tag{2.16}$$

**Example 8**

Based on the following information for a life table: $l_{20}$ = 83,333 and De Moivre model is assumed with ω = 120.

$$S_x(t) = 1 - \frac{t}{\omega - x} = 1 - \frac{30}{120 - 20} = \frac{70}{100} = 0.70 .$$

This means that the probability that an individual of age 20 years will survive 30 years is 70%. Therefore from (2.16), the number of survivors at age 50 is given as, $l_{50}$ = 83,333 × 0.70 = 58,333.

We conclude this section with a consideration of the expected lifetime of an individual of age x years using the De Moivre model. This is a very important issue facing life insurance actuaries since it is required in estimating premiums for life insurance and life annuities.

### 2.2.4 COMPLETE EXPECTED LIFE

We want to predict how long an individual of age *x* years will live assuming that there is a maximum lifetime of ω years. In other words, for an individual of age *x* years we want to calculate the expected value of the individual's future lifetime – that is, $E(T_x)$. This is called the **complete expected life** for an individual of age *x* years.

Under De Moivre model which assumes that $T_x$ has a uniform distribution over the interval [x, ω], the expected value is simply the midpoint of this interval. Hence, $E(T_x) = \frac{\omega - x}{2}$.

The actuarial symbol for the complete expected life for an individual of age *x* years, is $\overset{o}{e}_x$. Hence,

$$\overset{o}{e}_x = \frac{\omega - x}{2} \tag{2.17}$$

**Example 9**

Assuming mortality is uniformly distributed over the interval [x, ω] where ω is the ultimate age of 120 years, an individual currently of age 50.3 years, is expected to live one-half of (120 – 50.3) = ½ × 69.7 years = 34.85 years.

**Comment**

Since ω is the maximum life at death, equation (2.17) states that a life (*x*) is expected to live one-half of his/her **maximum** remaining lifetime.

A closely associated concept is called the **curtate expected life** for an individual of age *x* years.

### 2.2.5 CURTATE EXPECTED LIFE

In Box 1.1 of Chapter 1, it is stated that the curtate future lifetime ($\mathbf{K_x}$) of an individual of age *x* years is the number of full or complete years that the individual has survived to date. It is the integer part of the remaining future lifetime random variable $\mathbf{T_x}$. Hence we want to calculate $\mathbf{E(K_x)}$.

Under De Moivre model of UDD, it can be shown that:

$$\mathbf{E(K_x) = \frac{\omega - x - 1}{2} = \frac{\omega - x}{2} - \frac{1}{2}} \qquad (2.18)$$

**Comment**

The actuarial symbol for the curtate expectation of life is $\mathbf{e_x}$. Substituting (2.17) into (2.18), we obtain:

$$\mathbf{E(K_x) = E(T_x) - \frac{1}{2}}.$$

Using actuarial notation: $\mathbf{e_x = \overset{o}{e}_x - \frac{1}{2}}$ (2.19)

**Example 10**

For the information in Example 9 and applying equation (2.18), $\mathbf{E(K_{50.3})} = \frac{120 - 50.3 - 1}{2} = \frac{68.7}{2} = 34.35$ years.

**Comment**

Although $\mathbf{K_x}$ is the integer part of $\mathbf{T_x}$, $\mathbf{E(K_x)}$ is not the integer part of $\mathbf{E(T_x)}$.

Another key concept in insurance is called the **force of mortality.**

To set the stage for a discussion of the **force of mortality**, we refer to the first eBook in Level 1 entitled *Principles of Insurance* where we considered the theory of interest and in particular, the **force of interest**. Force of interest may be explained intuitively as follows: assume that an investor has a sum of money accumulated at time **t.** Consider the increase in the accumulated sum of money over the next instant of time. The force of interest is defined as the instantaneous increase in sum of money at time **t** as a proportion of the accumulated sum of money at time **t**.

Another explanation is even more intuitive and is found in the Journal of the Institute of Actuaries, Volume 16, page 451 (January,1872). We paraphrase the logic of the explanation presented there.

Suppose that a person travels between points A and B covering a distance of one mile in 15 minutes. The average speed is 4 miles per hour. Of course, this person may not be travelling at a **constant** speed of 4 miles per hour. Sometimes he/she goes faster and other times travels slower than 4 mph. The **force of speed at A** (our phrase) is the speed when he/she just start to travel from A, that is, the instant after starting to travel from A.

By analogy, we define **force of mortality**.

### 2.2.6 FORCE OF MORTALITY

The concept of **force of mortality** is similarly defined. Consider an individual of age 40 years. What is the probability that this individual will die the next instant? This probability is called the **instantaneous rate of mortality or the force of mortality**.

Clearly, for an individual of 100 years, the force of mortality will be higher than at age 40 years.

The actuarial symbol for the force of mortality is $\mu_x$. For the De Moivre model, the force of mortality is the height of the rectangle described in Figure 2.5. The probability of an individual of age *x* years will die the next instant of time is the height of the rectangle, $\mu_x = \frac{1}{\omega - x}$.

**Summary**

The force of mortality is the instantaneous rate of death at age x years conditional that the individual survives up to age x years. For the De Moivre model, $\mu_x = \frac{1}{\omega - x}$, $\omega =$ **ultimate age**

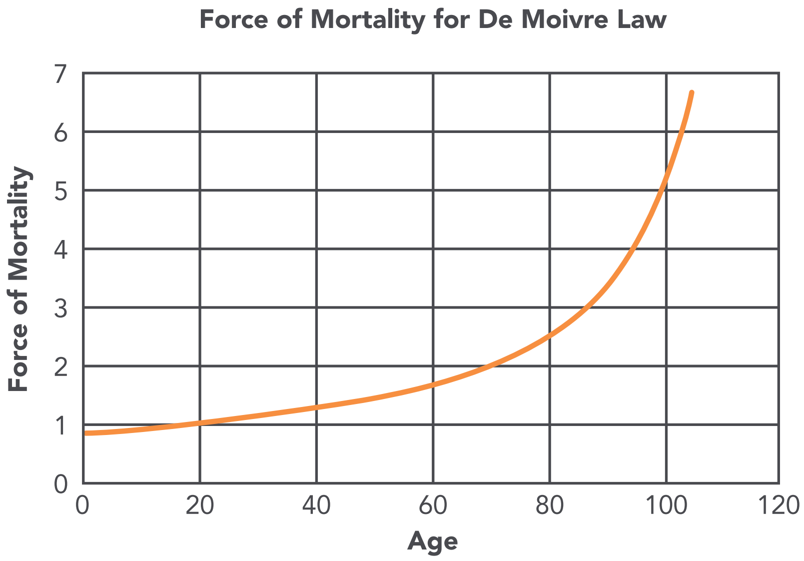
**Comment**

From the De Moivre model, the force of mortality increases with age with significant increases at older ages. For example, for ω = 120 years, and a life (50), the force of mortality $\mu_x = \frac{1}{120-50}$ = 0.014286=1.4286%. This is the probability that an individual of age 50 years will die the next instant of time.

In the case of (100), $\mu_x = \frac{1}{120-100}$ = 5% which is the instantaneous mortality rate that is much that for a life (50).

This trend in the force of mortality is observed in graph below of the force of mortality (in %) for the De Moivre model against different ages where ω =120 years.

| | |
|---|---|
| 0 | 0.833333 |
| 5 | 0.869565 |
| 10 | 0.909091 |
| 15 | 0.952381 |
| 20 | 1 |
| 25 | 1.052632 |
| 30 | 1.111111 |
| 35 | 1.176471 |
| 40 | 1.25 |
| 45 | 1.333333 |
| 50 | 1.428571 |
| 55 | 1.538462 |
| 60 | 1.666667 |
| 65 | 1.818182 |
| 70 | 2 |
| 75 | 2.222222 |
| 80 | 2.5 |
| 85 | 2.857143 |
| 90 | 3.333333 |
| 95 | 4 |
| 100 | 5 |




To this point in the chapter, we considered life tables which typically provide survival and mortality probabilities for *integer ages and integer duration.* But it may be required to calculate mortality probabilities for individuals with fractional ages (e.g., (40.5)) or for a duration less than one year. We now discuss the issues involved in fractional age in life tables.

## 2.3 FRACTIONAL AGE IN LIFE TABLES

Obtaining mortality probabilities for fractional ages requires an assumption of the distribution of deaths between integer ages; that is, between x and x+1. A simple approach in dealing with fractional ages is called *linear interpolation* that assumes *uniform distribution of deaths (UDD).*

### 2.3.1 LINEAR INTERPOLATION ASSUMPTION (UDD)

We first consider the case of an integer age and a fractional time period.

**First Case: Integer Age and Fractional Duration**

We motivate this case by means of an example and then provide details of the methodology.

**Example 11**

Refer to the illustrative life table provided above (Table 2.1). For age 50 years, the number of individuals living up to age 50 years is 89,509 and the number living up to age 51 years is 88,979. This means that there are 530 deaths between ages 50 years and 51 years. From this information, we calculate that the mortality rate at age 50 years is: $q_{50} = \frac{l_{50} - l_{51}}{l_{50}} = \frac{530}{89{,}509} =$ 0.005921.

*This is probability that an individual of age 50 years will die one complete year later.*

How do we answer the question: what is the probability that an individual of age 50 years will die a fraction of a year later – e.g., on or before 50.25 years?

The answer is provided through what is commonly called a linear interpolation approach. An illustration of the assumption of linear interpolation is shown in the next diagram:


Notes:
a) *0 < s <1*; that is, *s is a fraction*.
b) Linear interpolation implies that the number of deaths in the period [x, x+1] is shared in proportion to the values of *s*.

**Figure 2.6:** Linear Interpolation (Uniform Distribution of Deaths, UDD)

For the case of an individual aged 50.25 years, $s$ = 0.25. Under the assumption of uniform distribution of deaths, since there are 530 deaths for the full year, then we assign one quarter of 530 for one-quarter of the year. Hence 132.5 deaths are assigned for the period between 50 years and 50.25 years. Since $l_{50}$ = 89,509, then $l_{50.25}$ = *89,509 – 132.5 = 89.376.5*

Therefore, the probability of a life of age 50 years dying before 0.25 year or at age 50.25 years, is given as follows:

$$ {}_{0.25}q_{50} = \frac{l_{50} - l_{50.25}}{l_{50}} = \frac{89{,}509 - 89{,}376.5}{89{,}509} = \frac{132.5}{89{,}509} = 0.00148 $$

**Comment**

Here is an interesting insight: As we showed above in example 11, for an individual of age 50 years, the probability of dying over the next year is $q_{50}$ = 0.005921. It was also shown that for an individual of age 50 years, the probability of dying over the next one quarter of a year is ${}_{0.25}q_{50}$ =0.00148.

**Note that 0.00148 is exactly one quarter of 0.005921!**

The probability of death for one quarter of a year is exactly one quarter the probability for the full year.

This is an example of a general and important formula:

$$ {}_{s}q_{x} = sq_{x},\ 0 < s < 1 \tag{2.20}$$

Returning to example 11, $q_{50} = 0.005921$ and $s=0.25$. Therefore, ${}_{0.25}q_{50} = 0.25 \times q_{50}$ $= 0.25 \times 0.005921 = 0.00148$.

Our second case involves, in addition, a fractional age.

**Second Case: Fractional Age and Fractional Duration**

The following diagram illustrates the second case:

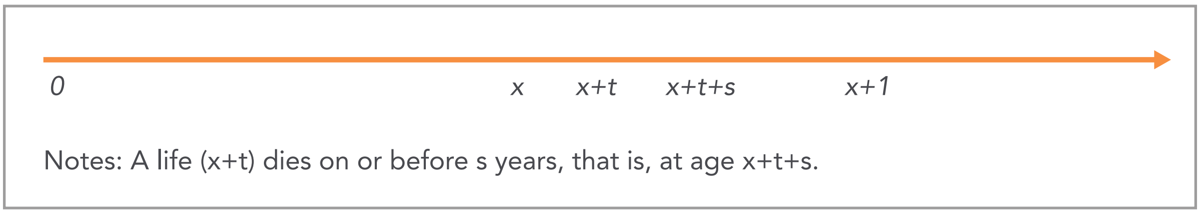



**Figure 2.7:** Linear Interpolation (Uniform Distribution of Deaths, UDD)

An example that illustrates a fractional age and a fractional duration is for an individual of age 50.60 years and requiring the probability that he/she dies before 0.25 year – that is, ${}_{0.25}q_{50.60}$. With reference to Figure 2.7, *x= 50, t=0.60 and s= 0.25.*

The main result is as follows:

$$ {}_{s}q_{x+t} = \frac{s \times q_{x}}{1 - t \times q_{x}}; \qquad 0 \le s \le s + t \le 1 \tag{2.21}$$

**Note that x is an integer age. If t=0, we have the same result as equation (2.20).**

Here is an example illustrating (2.18).

**Example 12**

From the illustrative life table in Table 2.1, we calculate $q_{50}$ = 0.005921 (see example 11). From equation (2.21), $_{0.25}q_{50.60} = \frac{0.25 \times 0.005921}{1 - 0.60 \times 0.005921}$ = 0.001486.

**Comment**

Equation (2.21) can be restated as follows: $_{s}q_{x+t} = \frac{s}{\left( 1/q_x - t \right)}$ (2.22)

Return to example 11 and using (2.22), we obtain: $_{0.25}q_{50.60} = \frac{0.25}{\left( 1/0.005921 - 0.60 \right)} = 0.001486$ as found in example 12.

This concludes this chapter. Chapter 3 considers premiums for life annuities.

# 3 PREMIUMS FOR LIFE ANNUITIES

## 3.1 INTRODUCTION

Formally, a life annuity is a contract where an individual of age $x$ years receives periodic payments (e.g., monthly or annually) from an insurance company in exchange for a single premium paid to the insurance company at the beginning of the contract. The key point is that the periodic payments are **contingent** on the survival of the annuitant.

If payments are made at the beginning of each period (year or month) for as long as the individual is alive, the life annuity is called a **whole life annuity due**. In the case of a w**hole life annuity immediate**, payments are made at the end of each period contingent on the survival of the annuitant.

For a t**emporary life annuity due**, the contract matures in $n$ years meaning that the annuitant receives payments at the beginning of each period for a term no longer than $n$ years. In addition, payments cease if the annuitant dies within $n$ years. We also present the single premium of a **deferred whole life annuity due**. Interestingly, a whole life annuity is equal (in fundamental terms) to a combination of a temporary life annuity due and a deferred whole life annuity due. This is a useful relationship in that it can be used to deduce single premiums for some life annuities.

We note that the annuitant is exposed to the risk of dying earlier than expected and has the risk of receiving fewer payments than expected. As stated in chapter 1, the annuitant may mitigate this risk by investing in a whole life annuity due with a guarantee of $n$ payments for a specified number of years. This insurance product is called an **n-year period certain whole life annuity due** in actuarial terminology. We show that the single premium for this life annuity due is the sum of the single premiums for a certain annuity due and a deferred whole life annuity due. This result will facilitate an easy deduction of single premiums of period certain whole life annuity due.

**Comment**

This chapter considers life annuities due in relative detail since payments to the beneficiary are made at the beginning of the period and so are appropriate for retirement plans. We will make brief comments on corresponding life annuities immediate at appropriate points in this chapter.

### An Intuitive Approach

The calculation of single premiums of life annuities typically involves a high level of mathematical methods. We choose a discovery approach through a case study assuming that mortality follows the illustrative life table (ILT) and a technical interest rate of 6% that was introduced in chapter 2. This greatly simplifies calculations and enhances clarity of the main principles. The general formulae for single premiums of life annuities are provided in Box 3.1 below.

## 3.2 WHOLE LIFE ANNUITY DUE

### Case Study: Choosing a Life Annuity Due Upon Retirement

***Part A***

Mario Perez has just retired at the age of 70 years. He does not want to take on the risk of outliving his financial resources and decides to purchase a whole life annuity due from his insurance company using his savings accumulated over his working life. He visits Bob Rosen, the company's sales and customer relations professional.

Based on available demographic information for Mario, the actuary of the insurance company predicts that Mario's remaining lifetime is 12.6 years; equivalently, his predicted age at death is 82.6 years. Mario wants to know the amount he must pay today in exchange for annual payments of €50,000 at the beginning of each year for his future lifetime.

Based on this information, Rosen advises Mario that the single premium for the whole life annuity due is €373,965.

Rosen proceeded to explain the details on how the single premium was obtained. He begins by explaining the following diagram where each annual payment is €1 paid at the beginning of each year.


**Figure 3.1:** Illustration of a Whole Life Annuity Due

Rosen explains that Figure 3.1 is based on the fact that Mario is of age 70 years with a predicted future lifetime of 12.6 years and hence the number of *complete* years that Mario is expected to live is 12 years. Hence, Mario is expected to receive 13 payments.

Rosen explains that the single premium is obtained by assuming mortality follows the ILT using an interest rate of 6%. The data for survivors over time is presented as follows:

| Age (x) | 70 | 71 | 72 | 73 | 74 | 75 |
|---|---|---|---|---|---|---|
| $l_x$ | 66,161 | 63,966 | 61,647 | 59,204 | 56,640 | 53,961 |

| Age (x) | 76 | 77 | 78 | 79 | 80 | 81 | 82 |
|---|---|---|---|---|---|---|---|
| $l_x$ | 51,171 | 48,282 | 45,304 | 42,252 | 39,144 | 36,000 | 32,845 |

The formula used to calculate the single premium for an annual payment of €1 and using the data for Mario is:

$$\boldsymbol{SP} = \mathbf{1} + \left(v \times \frac{l_{x+1}}{l_x}\right) + \left(v^2 \times \frac{l_{x+2}}{l_x}\right) + \ldots + \left(v^{12} \times \frac{l_{x+12}}{l_x}\right) \quad (3.1)$$

**SP** = single premium and is represented by the actuarial symbol $\ddot{a}_x$.

Using equation (3.1), for the case where $x = 70$ and the data from the ILT presented above, the single premium is obtained as follows:

$$\ddot{a}_{70} = 1 + \left(0.9434 \times \frac{63{,}966}{66{,}161}\right) + \left(0.9434^2 \times \frac{61{,}647}{66{,}161}\right) + \ldots + \left(0.9434^{12} \times \frac{32{,}845}{66{,}161}\right) = 7.4793$$

Rosen explains that:

a) The single premium for an annual payment of €1 at the beginning of each year for an individual of age 70 years who is expected to survive until age 82.6 years is €7.4793.
b) Since Mario wants €50,000 at the beginning of each year, then the single premium in a) is multiplied by €50,000 to obtain €373,965.

**Comment**

Rosen is aware that the insurance company is taking the risk that Mario may live longer than the actuary predicts and so he may collect more than the expected thirteen annual payments each of € 50,000.

As a default position, the insurance company using the ILT for a maximum lifetime of $\omega$ =110 years usually calculates the single premium using equation (3.1). This alternative obtains a value of $\ddot{a}_x$ =8.5693. This value is 14.6% higher than the single premium obtained when the predicted remaining lifetime is 82.6 years. The single premium is therefore €428,465 for the same annual payment of €50,000 based on the ultimate age of 110 years. Rosen understands that while this higher single premium helps to mitigate the risk faced by the insurer, it may result in a non-competitive single premium and the potential for a loss of business.

*This completes Part A of the case study and we generalise equation (3.1) that is included in Box 3.1 below.*

**Box 3.1: General Case for the Single Premium of a Whole Life Annuity Due**

**Note:**
In the actuarial literature, the formula for the single premium of a whole life annuity due is based on the annuitant's future lifetime being infinite. This is theoretically correct. But life insurance actuaries typically predict the expected curtate future lifetime of an annuitant (as in Mario's case) or use life tables such as the ILT model with a finite ultimate age. Accordingly, we present two cases below which adopt these pragmatic approaches and do not consider the case when age approaches infinity.

**Single Premium for Whole Life Annuity Due**
*(First Case: Based on an Actuary's Prediction of Annuitant's Future Lifetime)*

We note that in this case, the number of payments received by the annuitant for a whole life annuity due is $K_x$ +1 where $K_x$ is the annuitant's predicted curtate future lifetime (i.e., in complete years).

The single premium for a whole life annuity due is:

$$\ddot{a}_x = 1 + \left(v \times \frac{l_{x+1}}{l_x}\right) + \left(v^2 \times \frac{l_{x+2}}{l_x}\right) + \ldots + \left(v^{K_x} \times \frac{l_{x+K_x}}{l_x}\right) \qquad (3.2)$$

This formula is called the *Current Payment Technique Formula*

**Comment**

The risk for the insurer is that the annuitant lives longer than the predicted remaining lifetime and having to make more than $K_x$ +1 payments to the annuitant.

The eBook entitled *Risk Management* in Level II of the Art of Insurance series considers this issue, called **longevity risk**. At this time, we note that from a portfolio perspective, the insurer can mitigate this risk through the *principle of mutuality* whereby annuitants who die earlier than predicted subsidise those who survive longer than expected. There are other market-based approaches in Level II to hedge this risk.

b) **Single Premium for Whole Life Annuity Due**
*(Second Case: Based on the Ultimate Age in the ILT)*

The general formula in this case is as follows:

$$\ddot{a}_x = 1 + \left(v \times \frac{l_{x+1}}{l_x}\right) + \left(v^2 \times \frac{l_{x+2}}{l_x}\right) + \ldots + \left(v^{\omega} \times \frac{l_{x+\omega}}{l_x}\right) \quad (3.3)$$

Note that $\omega$ is the ultimate age in the life table.

Before we proceed to Part B of the case study, we consider two special cases. These are the *single premium of an n-year temporary life annuity due* and of a *whole life annuity deferred for n years.*

### 3.2.1 TEMPORARY LIFE ANNUITY DUE

In the case of a temporary life annuity, the term is selected by the annuitant at the start of the contract. Specifically, the insurer pays a benefit of €1 for n years at the beginning of each year t=0, 1, 2, …, n-1. The actuarial symbol for the single premium for an *n*-year temporary life annuity due is $\ddot{\mathbf{a}}_{\mathbf{x:\overline{n}|}}$.

As an example, consider the single premium of a 5-year temporary life annuity due for $x = 70$ and $n = 5$. From equation (3.1), we obtain:

$$\ddot{a}_{70:\overline{5}|} = 1 + \left(0.9434 \times \frac{63{,}966}{66{,}161}\right) + \left(0.9434^2 \times \frac{61{,}647}{66{,}161}\right) + \left(0.9434^3 \times \frac{59{,}204}{66{,}161}\right) + \left(0.9434^4 \times \frac{56{,}640}{66{,}161}\right) \quad (3.4)$$

$= 4.1708$

For a payment of €50,000, the single premium is 4.1708 × €50,000 = €208,540.

### 3.2.2 N-YEAR DEFERRED WHOLE LIFE ANNUITY DUE

The second special case is a whole life annuity due that is deferred for *n* years meaning that the first payment is made at the beginning of the year, *n* years from today. The actuarial symbol is ${}_{n|}\ddot{a}_x$. We consider the single premium of the whole life annuity due is deferred for five years as follows:

For x=70 years and n=5, the first payment occurs at the beginning of the year from 70+5 = 75 years and continues for a predicted total number of eight payments. To obtain the single premium in this case, we calculate ${}_{5|}\ddot{a}_{70}$ based on the following formula:

$${}_{5|}\ddot{a}_{70} = v^5 \times \frac{l_{75}}{l_{70}} + v^6 \times \frac{l_{76}}{l_{70}} + \ldots + v^{12} \times \frac{l_{82}}{l_{70}} \quad (3.5)$$

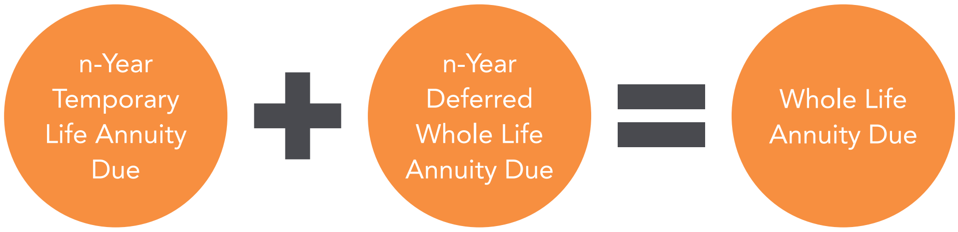
Using the data from the ILT with using equation (3.5) we have:

$${}_{5|}\ddot{a}_{70} = \left(0.9434^5 \times \frac{53{,}961}{66{,}161}\right) + \left(0.9434^6 \times \frac{51{,}171}{66{,}161}\right) + \left(0.9434^7 \times \frac{48{,}282}{66{,}161}\right) + \ldots + \left(0.9434^{12} \times \frac{32{,}845}{66{,}161}\right)$$

$= 3.3085$

**Important Relationship for Annuities**

These examples illustrate an interesting general result: The single premium for the whole life annuity due (€7.4793) is equal to the single premium for a 5-year temporary life annuity due (€4.1708) and a 5-year deferred whole life annuity due (€3.3085). This relationship is shown in the diagram below:




We now continue with the case study and consider Part B.

Mario contemplates the size of the single premium obtained in Part A and concludes that if he dies soon after signing the annuity contract, he would pay a relatively large single premium in exchange for only a few annual payments.

***Part B:***

To reduce his risk from dying prematurely, Mario chooses five guaranteed annual payments of €50,000 which are payable even if Mario dies at any time earlier than five years. In addition, payments continue beyond five years provided Mario is still living. These terms and conditions describe a **5-year period certain whole life annuity due** which has an actuarial symbol, $\mathbf{a}_{\overline{x:\overline{n}|}}$.

Rosen explains to Mario that a 5-year period certain whole life annuity due is equivalent to two insurance products.

The first product is a **certain annuity due** which does not depend on the survival of Mario during the first 5 years of retirement. Five payments will be made by the insurance company with certainty. The second product is a 5-year **deferred** whole life annuity due from the start of year 5 and end on year 12 for a total of 8 payments. These eight payments are contingent on the survival of Mario.

Rosen obtains the single premium for a **certain annuity due** for n = 5 as follows:

$$\ddot{a}_{\overline{n}|i} = \frac{1-v^5}{d}; \quad v = \frac{1}{1.06} = 0.9434; \ d = \frac{0.06}{1.06} = 0.056604 \tag{3.6}$$

Hence, the single premium for this product is: $\ddot{a}_{\overline{n}|i} = \frac{1-v^5}{d} = \frac{1-(0.9434)^5}{0.0566404} = 4.4648$ (3.7)

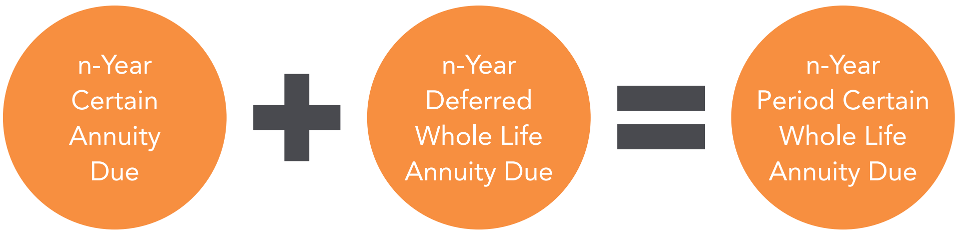
Rosen noted that the single premium for a 5-year deferred whole life annuity due is €3.3085.

Hence the total single premium for this 5-year period certain whole life annuity due is found by adding the single premium for each of the two components giving a value of €4.4648+€3.3085 = €7.7733 for an annual payment of €1. Multiplying by €50,000 we obtain a single premium €388,665.

Rosen explains that the single premium for a straight whole life annuity due is €373,965. For a 5-year period certain whole life annuity, the single premium is almost 4% higher.

**Important Relationship for Life Annuities**

This example illustrates as key point: The single premium for an *n*-year period certain whole life annuity due is the sum of the single premium for an *n*-year certain annuity due and the single premium of corresponding *n*-year deferred whole life annuity due. This relationship is shown in the diagram below:




The formulae applied in this case study are now generalised in Box 3.2 below

**Box 3.2: Formula for Period Certain Whole Life Annuity Due**

There are two components involved in procedure to obtain the single premium for an n-year Period Certain Whole Life Annuity Due.

The first component is the single premium for an n-year certain annuity due, $\ddot{a}_{\overline{n}|i} = \dfrac{1-v^n}{d}$ and the second component is the single premium for a whole life due deferred for n years. In this case, the formula is as follows.

$${}_{n|}\ddot{a}_x = v^n \times \frac{l_{x+n}}{l_x} + v^{n+1} \times \frac{l_{x+n+1}}{l_x} + \dots + v^{K_x} \times \frac{l_{K_x}}{l_x}$$

Hence the n-year period certain annuity is equal to: $a_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|i} + {}_{n|}\ddot{a}_x$

To this point in our presentation, we show how to calculate the single premium for life annuities due based on annual payments of monetary value 1. However, it is common practice for payments to the annuitant to be made on a monthly basis – that is, equal payments made 12 times a year.

The actuarial terminology for equal payments made **m** times a year is $a_x^{(m)}$. For example, $a_{70}^{(12)}$ represents the single premium for a whole life annuity for an individual of age 70 years where payments are made by the insurer at the beginning of each month.

The Woolhouse formula (for 2 terms) that is based on the assumption of uniform distribution of deaths (UDD) provides a relationship between annual payments and payments made **m** times a year for a whole life annuity due. For small interest rates, this formula is as follows:

$$a_x^{(m)} = a_x - \frac{m-1}{2m} \tag{3.8}$$

Recall that for Mario, the single premium for a whole life annuity due based on annual payments is equal to 7.4793. Then the single premium for the identical whole life annuity but for monthly payments is given by: $a_{70}^{(12)} = a_{70} - \dfrac{12-1}{2\times 12} = 7.4793 - \dfrac{11}{24} = 7.0210$.

**Comment**

It is reasonable that for a whole life annuity due, the single premium for monthly payments is less than that for annual payments. This is because higher compounding frequency leads to a lower present value for a fixed future payment.

Equation (3.8) shows that as **m** increases, the factor $\dfrac{m-1}{2m} \Rightarrow \dfrac{1}{2}$ and so $\ddot{a}_x^{(m)} = \ddot{a}_x - \dfrac{1}{2}$.

There are similar Woolhouse (2-term) formulae for typical life annuities. For an n-year temporary life annuity and small interest rates, the formula for m payments during the year is:

$$\ddot{a}^{(m)}_{x:\overline{n}|} = a_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n \times \frac{l_{x+n}}{l_x}\right) \tag{3.9}$$

We show the calculation of the single premium for a 5-year temporary life annuity due for monthly payments (m=12).

In section 3.1.1, for x=70 years, n=5 years, we showed that the single premium for annual payments is 4.1708. From the Illustrative Life Table shown in section 3.1 for Mario, we have the following:

$l_{70} = 66{,}161; \quad l_{75} = 53{,}961; v = 0.9434$. Substituting into (3.9) we obtain:

$$\ddot{a}^{(12)}_{70:\overline{5}|} = 4.1708 - \frac{11}{24}(1 - 0.6064) = 3.9904$$

The single premium is 3.9904 for monthly payments corresponding to an annual payment of 1.

For Mario's required €50,000 annual payment, the single payment for equivalent monthly payments is €199.520 compared to €208,540 when payments were made annually.

We close this chapter with a description of the relationship between life annuity due and life annuity immediate.

**Linking Life Annuities Due and Immediate**

a) For an n-year life annuity immediate, the insurer makes periodic payments *at the end of the period* contingent on the survival of the annuitant. For an n-year whole life annuity immediate, the actuarial symbol is $a_x$. The actuarial symbol for a life annuity immediate is similar to that for a life annuity due **except** without the *trema.* The formula for the single premium for a whole life annuity immediate is:

$$a_x = \left( v \times \frac{l_{x+1}}{l_x} \right) + \left( v^2 \times \frac{l_{x+2}}{l_x} \right) + \ldots + \left( v^{K_x} \times \frac{l_{x+K_x}}{l_x} \right)$$

Comparing for the corresponding formula for a whole life annuity due in (3.2) we see that:

$$\ddot{a}_x = a_x + 1 \tag{3.10}$$

For Mario who is (70), the single premium for a whole life annuity due making annual payments is shown above as 7.4793. The corresponding single premium for a whole life annuity immediate is $a_{70}$ = 7.4793 -1 = 6.4793. The single premium is reduced by a single annual payment compared for the while life annuity due.

Similarly for an n – year temporary life annuity immediate we have that:

$$\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 \tag{3.11}$$

For example, we showed in section 3.1.1 that for Mario who is (70), the single premium for a 5-year temporary life annuity due with annual payments is 4.1708. The corresponding single premium for a 5-year temporary life annuity immediate for monthly payments is $a_{70:5} = 4.1708 - 1 = 3.1708$.

**Comment**:
Equations (3.1) and (3.11) are statements where payments to the annuitant are annual. They also hold when payments are made more than once during the year.

This concludes chapter 3. The next chapter considers the actuarial valuation of life insurance and establishes a close relationship between annuities and their counterparts in life insurance.

# 4 PREMIUMS FOR LIFE INSURANCE

## 4.1 INTRODUCTION

This chapter deals with the calculation of premiums for life insurance that is based on the equivalence principle. A detailed discussion of principles of premium calculation is presented in the first eBook entitled *Principles of Insurance.* For completeness, we provide a brief review of the main issues.

Under the equivalence principle, the premium is set such that at the start of the contract the expected value of the future loss is zero. This implies that the expected present value of premiums paid to the insurer is equal to the expected present value of benefits paid by the insurer to the insured.

Alternatively, at the date of policy issue the actuarial present value of current and future premiums is equal to the actuarial present value of future benefits. The premium so obtained is called the **pure premium**. Pure premiums are premiums determined by the equivalence principle, ignoring expenses or profit margins that the insurer may add on.

The key point is that *the premium (set according to the equivalence principle) will be sufficient to cover the average loss per contract at the date of policy issue.*

We now operationalise the formula for single premium and infer annual level premiums for life insurance from single premiums of appropriate life annuities considered in chapter 3.

## 4.2 PREMIUMS FOR LIFE INSURANCE

As noted in Chapter 1, the two fundamental life insurance contracts are the pure endowment insurance and term life insurance. These life insurance contracts are the building blocks on which others are created. We begin with the procedure for calculating the premiums for these contracts and then follow with special cases of whole life and endowment insurance.

Here is an important assumption that underlines the presentation in this chapter and that serves to maintain simplicity and enhance understanding of the key issues and principles.

**Assumption (fully discrete insurance)**

The benefit payment is assumed to made at the *end of the year of death.* The advantage is that we can utilise the information for mortality tables using a life table. As in Chapter 3, we will use the ILT model with an assumed interest rate of 6%.

In addition, annual premiums are discrete and paid to the insurer at the beginning of each year and so have similarities to life annuities due. Hence the calculation approach used in chapter 3 will be repeated in this chapter especially for level annual premium payments due, for example, monthly.

### 4.2.1 PURE ENDOWMENT INSURANCE

**Case of a Single Premium**

A pure endowment insurance pays a fixed benefit of B=1 at the end of the *nth* year if the policyholder survives at least *n* years.

For reference, time 0 is the date the insurance contract is issued. The most common premium principle in life insurance is the equivalence principle which states that the single premium at time 0 equals the expected present value of the benefit payment. Figure 4.1 illustrates this premium calculation principle and explanations follow:


**Figure 4.1** Pure Endowment Insurance ( $\Pi_0$ is the single premium)

The formula for the single premium based on the equivalence principle states that the single premium at time 0 is the expected present value of the benefit payment (B). The calculation takes two steps:

*First Step: (Calculate the Present Value of the Benefit)*

Calculate the present value of the benefit B=1 which is equivalent to the discounted value of 1 for *n* years and constant interest rate (*i*). Therefore, $\mathbf{PV(B=1)=v^n;\ v=\frac{1}{1+i}}$.

*Second Step: (Calculate the Expected Present Value)*

Multiply the result in the first step by the probability that an individual of age *x* years will survive at least *n* years, $_n p_x$. Using the ILT, $_n p_x = \frac{l_{x+n}}{l_x}$

Therefore, the single premium ($\Pi_0$) for a pure endowment insurance paying 1 is given in (4.1) below:

$$\Pi_0 = v^n \times {}_n p_x \quad (4.1)$$

**Comment**

In actuarial notation, **E** represents the present value of an endowment. Accordingly, the actuarial notation for the **single premium** of an n-year pure endowment insurance paying 1 for a policyholder of age *x* years is $_n E_x$. Hence equation (3.1) is stated as follows:

$$_n E_x = {}_n p_x \times v^n \quad (4.2)$$

**Example 1 (Using De Moivre mortality model)**

Given the following information:

Policyholder of age 50 years; 10-year pure endowment insurance with a benefit of €1; interest rate is 6%; survival probability is determined by the De Moivre model where ω=120 and $_n p_x = 1 - \frac{n}{\omega - x}$.

Therefore, the single premium for this insurance according to the equivalence principle is obtained as follows: $_{10} p_{50} = 1 - \frac{10}{120 - 50}$ = 0.8571; also $v^{10} = \left(\frac{1}{1.06}\right)^{10}$ = 0.5584.

Using (4.2), the single premium for this n-year pure endowment insurance is equal to:

$_{10} E_{50} = {}_{10} p_{50} \times v^{10} = 0.8571 \times 0.5584 =$ 0.4786 = 47.86%. The single premium is 47.86% of the benefit value.

Here is another example for the single premium of a pure endowment insurance where the interest rate is lowered from 6% to 3%.

**Example 2 (Using De Moivre mortality model; lower interest rate)**

A life (50) purchases a 10-year pure endowment insurance for €50,000. Assume interest rate of 3% so that $\mathbf{v} = \frac{1}{1.03} = 0.9709$ and De Moivre model of mortality for ultimate age of 120.

The single premium for this insurance for a benefit of €1 according to the equivalence principle is obtained as follows:

$$ {}_{10}E_{50} = {}_{10}p_{50} \times v^{10} = \left(1 - \frac{10}{120-50}\right) \times (0.09709)^{10} = 0.8571 \times 0.7441 = 0.6378. $$

This is the single premium which is 63.78% of a benefit value of B = €1.

**Comment**

All else equal, a lower interest rate results in a higher present value and so a higher single premium. With a 6% interest rate, we obtained a single premium of 47.86% of a benefit value of €1. For the same benefit, example 2 with a lower interest rate of 3% results in 63.78%.

Here is a third example based on the ILT in Table 2.1 of Chapter 2. We note that using life table data and equation (2.5) in Chapter 2, formula (4.2) becomes, ${}_nE_x = \frac{l_{x+n}}{l_x} \times v^n$ (4.3)

**Example 3 (Using ILT model)**

A life (70) purchases a 5-year pure endowment insurance. Based on data from the ILT model and interest rate of 6%, what is the single premium for this insurance?

From the ILT, $l_{70} = 66{,}161$; $l_{75} = 53{,}961 \Rightarrow {}_5p_{70} = \frac{l_{75}}{l_{70}} = \frac{53{,}961}{66{,}161} =$ 0.8155.

Also $v^5 = \left(\frac{1}{1.06}\right)^5$ = 0.7472. So ${}_5E_{70} = 0.7472 \times 0.8155 =$ 0.6093 or 60.93% of the benefit payment.

We now consider the actuarial valuation of term life insurance contracts where benefit payments are made at the end of the year of death.

### 4.2.2 TERM LIFE INSURANCE CONTRACTS

**Case of a Single Premium**

We begin with an intuitive discussion to discover the main issues involved in obtaining a single premium for a term life insurance. Here is an example that facilitates our discussion.

Suppose a policyholder currently of age 70 years purchases a 5-year term life insurance. What is the single premium for this insurance policy?

We present an intuitive discovery process that will lead to a simple formula for the single premium.

For a 5-year term life insurance, there are five future outcomes stated as follows:

A: The policyholder dies during the first year of the policy. A payment of a sum insured of 1 is paid by the insurer to the beneficiary at the end of the first year. The present value of the sum insured is $v = \frac{1}{1+i}$

B: The policyholder survives the first year and dies during the second year. A payment of a sum insured of 1 is paid by the insurer to the beneficiary at the end of the second year. The present value of the sum insured is $v^2$.

***Observe the importance of the role of deferred mortality in this case. We introduced this concept in Figure 2.1 of Section 2.11 in Chapter 2.***

C: The policyholder survives the first two years and dies during the third year. A payment of a sum insured of 1 is paid by the insurer to the beneficiary at the end of the third year. The present value of the sum insured is $v^3$.

D: The policyholder survives the first three years and dies during the fourth year. A payment of a sum insured of 1 is paid by the insurer to the beneficiary at the end of the fourth year. The present value of the sum insured is $v^4$.

E: The policyholder survives the first four years and dies during the fifth year. A payment of a sum insured of 1 is paid by the insurer to the beneficiary at the end of the fifth year. The present value of the sum insured is $v^5$.

These five possible deferred mortality outcomes are mutually exclusive. Only one of these five outcomes can occur. Importantly, they are all based on the application of the concept of deferred mortality. We summarise each potential outcome in the following table:

| Five Possible Mortality Outcomes | Present Value of Sum Insured of 1 | Probability of Outcome (Refer to (2.7) of Chapter 2 using life table data) | Actuarial Present Value (Multiply column 2 by Column 3) |
|---|---|---|---|
| A | $v$ | $\frac{l_x - l_{x+1}}{l_x}$ | $v \times \frac{l_x - l_{x+1}}{l_x}$ |
| B | $v^2$ | $\frac{l_{x+1} - l_{x+2}}{l_x}$ | $v^2 \times \frac{l_{x+1} - l_{x+2}}{l_x}$ |
| C | $v^3$ | $\frac{l_{x+2} - l_{x+3}}{l_x}$ | $v^3 \times \frac{l_{x+2} - l_{x+3}}{l_x}$ |
| D | $v^4$ | $\frac{l_{x+3} - l_{x+4}}{l_x}$ | $v^4 \times \frac{l_{x+3} - l_{x+4}}{l_x}$ |
| E | $v^5$ | $\frac{l_{x+4} - l_{x+5}}{l_x}$ | $v^5 \times \frac{l_{x+4} - l_{x+5}}{l_x}$ |

Since the five possible outcomes are mutually exclusive, the single premium for this 5-year term life insurance is obtained by adding the values of the five cells in the last column in the table.

The actuarial symbol for the single premium of an n-year term life insurance is $A^1_{x:\overline{n}|}$. The formula for (x) is as follows:

$$A^1_{x:\overline{5}|} = v \times \left(\frac{l_x - l_{x+1}}{l_x}\right) + v^2 \times \left(\frac{l_{x+1} - l_{x+2}}{l_x}\right) + v^3 \times \left(\frac{l_{x+2} - l_{x+3}}{l_x}\right) + v^4 \times \left(\frac{l_{x+3} - l_{x+4}}{l_x}\right) + v^5 \times \left(\frac{l_{x+4} - l_{x+5}}{l_x}\right) \quad (4.4)$$

We illustrate this formula by an example.

**Example 4**

Policyholder is (70); 5-year term life insurance; interest rate is 6% so that v= 0.9434. Life data from the ILT is recorded in the table below:

| Age | 70 | 71 | 72 | 73 | 74 | 75 |
|---|---|---|---|---|---|---|
| $l_x$ | 66,161 | 63,966 | 61,647 | 59,204 | 56,640 | 53,691 |

Using (4.4) we obtain the single premium for this 5-year term insurance for (70) to be 0.1575 for a benefit payment of €1.

**Comment:**

This can be a prohibitive single premium for potential customers. For example, for a benefit payment of €100,000, the single premium is €15,750. Hence it is important to consider annual payments and then subsequently, monthly payments. We consider these approaches after we generalise the formula in (4.4) for an n-year term life insurance.

**Box 4.1: n-Year Term Life Insurance**

We generalise the previous example and consider and *n*-year term life insurance. The actuarial symbol for an *n*-year term life insurance as follows: $A^1_{x:\overline{n}|}$. This formula explains that the sum insured is 1 for the individual of age *x* years for a term of *n* years where payment is made at end of the year in which death occurs. The policy must be in-force. Hence (4.4) is generalised and simplified as follows:

$$A^1_{x:\overline{n}|} = \frac{v}{l_x} \times \begin{pmatrix} l_x + (v-1) \times l_{x+1} + (v^2 - v) \times l_{x+2} + (v^3 - v^2) \times l_{x+3} + \ldots + \\ (v^{n-1} - v^{n-2}) \times l_{x+n-1} - v^{n-1} l_{x+n} \end{pmatrix} \quad (4.5)$$

**The Case of Annual Premiums**

*We take the perspective of the policyholder of age x years who pays an annual premium of* $\boldsymbol{P_x}$ *at the start of each period but receives a benefit at the end of the year of death.*

The relationship between a single premium and annual level premiums paid at the start of each year for an n-year term insurance is as follows:

$$P_x \times \ddot{a}_{x:\overline{n}|} = A^1_{x:\overline{n}|} \quad \Rightarrow \quad P_x = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \tag{4.6}$$

**Comment**

Formula (4.6) shows the intrinsic link between life insurance and life annuities.
We illustrate by an example:

**Example 5**

For this example, we refer to example 4 where we calculated the single premium for a 5-year term life insurance. We obtained a single premium of 0.1575.

In section 3.1.2 of Chapter 3, we showed that a 5-year temporary life annuity due for (70) using the ILT is $\ddot{a}_{70:\overline{5}|}$ = 4.1708. Hence by (4.6) we obtain the annual level premium payable at the start of each year as, $P_x = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{0.1575}{4.1708} = 0.03776.$

This is equivalent annual premium for the single premium of 0.1575 for a sum insured equals to a monetary value of 1.

The fact that an endowment insurance is a combination of a term life insurance and pure endowment makes the calculation of its single premium very simple. This is shown next.

### 4.2.3 ENDOWMENT INSURANCE

**The Case of Single Premium**

Here is a diagram that highlights the simple procedure to obtaining the single premium for an endowment insurance:

| Insurance Type | Actuarial Symbol | Location of Formula in this Chapter |
|---|---|---|
| n-year Pure Endowment | $_{n}E_{x}$ | (4.1) |
| n-year Term Life | $A^{1}_{x:\overline{n}\vert}$ | (4.5) |
| n-year Endowment | $A_{x:\overline{n}\vert}$ | Equals to (4.1) + (4.5) |

Here is an example illustrating the approach for calculating the single premium for a 5-year endowment insurance for a 70-year old policyholder.

**Example 6**

Policyholder is (70); 5-year term life insurance; interest rate is 6% so that v= 0.9434. Mortality follows the ILT.

From Example 4, we obtained that for (70), the single premium for this term life insurance is 0.1575 of the sum insured. That is, $A^{1}_{x:\overline{n}\vert}$ = 0.1575.

Example 3 shows that single premium for a 5-year pure endowment for (70) is $_{n}E_{x}$ which is equal to 0.6094.

The single premium for the corresponding 5-year endowment insurance is: |

$$A_{70:\overline{5}\vert} = 0.1575 + 0.6064 = 0.7639$$

**Interesting Relationship**

There is a relationship between the single premium of an endowment insurance and temporary life annuity due as follows:

$$A_{x:\overline{n}\vert} = 1 - \left(d \times \ddot{a}_{x:\overline{n}\vert}\right) \qquad (4.7)$$

For an individual of age $x$ years and n=5, we have from section 3.1.2 of chapter 3 that $\ddot{a}_{70:\overline{5}\vert}$= 4.1708. Also, $d = \frac{0.06}{1.06}$ = 0.0566. Hence $A_{x:\overline{n}\vert} = 1 - (0.0566 \times 4.1708)$=0.7639 which is identical to the answer for example 6.

### The Case of Annual Premium

The single premium for an n-year endowment insurance can be converted into annual premiums payable at the start of each year is given by the formula:

$$P_x = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \tag{4.8}$$

As an illustration of (4.7), example 6 shows that a 5-year endowment insurance for (70) is $\mathbf{A}_{70:\overline{5}|} = 0.7639$.

As indicated above, $\ddot{a}_{70:\overline{5}|} = 4.1708$. Using (4.8), the annual premium paid at the start of each year is $P_{70} = \frac{A_{70:\overline{5}|}}{\ddot{a}_{70:\overline{5}|}} = \frac{0.7639}{4.1708} = 0.183154$

Equivalently, equation (4.8) may be restated as follows:

$$P_x = \frac{d \times A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}} \tag{4.9}$$

As an illustration of (4.9), we obtain for an interest rate of 6% and so d=0.0566,

$$P_x = \frac{0.0566 \times 0.7639}{1 - 0.7639} = 0.1831 \text{ as found above.}$$

We now consider the case of whole life insurance where one has to consider all potential outcomes until the ultimate age in the life table. We assume that death is certain at the ultimate age. In addition, the sum insured is 1 paid at the end of the year of death.

## 4.2.4 WHOLE LIFE INSURANCE

### The Case of Single Premium

The actuarial symbol for the single premium of a whole life insurance is $A_x$. There is a close relationship between whole life annuities due and corresponding whole life insurance that is described as follows:

$$A_x = 1 - d \times \ddot{a}_x \tag{4.10}$$

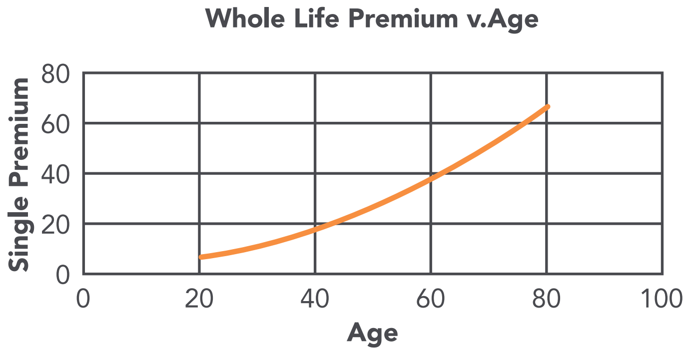
Referring to Box 3.1 of Chapter 3, the whole life annuity due, $a_x$ is equal to 8.5693 for (70).

From equation (4.10), we obtain a value for the single premium for whole life insurance for (x) as follows:

$$A_{70} = 1 - \frac{0.06}{1.06} \times 8.5693 = 0.5149$$

We repeat this calculation for selected ages and plot a graph of whole life insurance single premiums against age below:

| Age | Single Premium (%) |
|---|---|
| 20 | 6.528 |
| 30 | 10.248 |
| 40 | 16.132 |
| 50 | 24.905 |
| 60 | 36.913 |
| 70 | 51.495 |
| 80 | 66.575 |
| | |
| | |
| | |
| | |
| | |




The graph shows that as age increases, whole life insurance single premiums increase steadily and more sharply at later ages where the force of mortality is relatively higher.

## The Case of Annual Premium

Single premiums for whole life insurance can be converted into annual premiums paid at the beginning of each year as follows:

$$P_x = \frac{d \times A_x}{1 - A_x} \qquad (4.11)$$

where $P_x$ is the annual premium; $A_x$ is the single premium and $d = \frac{i}{1+i}$.

**Example 7**

In the previous table, it is shown that for an individual aged 70 years, the single annual whole life insurance premium based on data from the illustrative life table is $\boldsymbol{A_{70}}$ = 0.5149. For interest rate of 6%, **d**=5.6604%.

Formula (4.11) shows that $P_{70} = \frac{d \times A_{70}}{1 - A_{70}} = \frac{0.0566 \times 0.5149}{1 - 0.5149}$ = 0.0600 = 6.00% of the benefit payment.

Alternatively, (4.11) may be reformulated as: $P_x = \frac{A_x}{\ddot{a}_x}$ . (4.12)

Hence, $P_x = \frac{A_x}{\ddot{a}_x} = \frac{0.5149}{8.5695}$ = 0.0600= 6.00%.

To this point in the chapter, we have shown the procedure to calculate single and annual premiums for life insurance contracts. We conclude this chapter by showing the calculation procedure for the true level annual payment payable at the beginning of each month of the year.

Specifically, the benefit payment is made at the end of the year of death and premium payments are made **m** times during the year but at the beginning of each period. For example, if **m** is equal to 12, then payments are made at the start of each month.

Refer to example 7 where for an individual (70), the single premium for a whole life insurance is 0.5149 for a benefit payment at the end of year of death is 1. The corresponding annual payment is 0.0600. We want to calculate the true annual level premium that is payable monthly (m=12).

In this case, equation (4.12) is modified as follows:

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{m}}; \quad \ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} \quad (Woolhouse) \qquad (4.13)$$

**Example 8**

Consider the following information: Individual (70); whole life insurance where mortality follows ILT; Benefit Payment is $100,000. Calculate the true level annual premium payable monthly.

$A_x = 0.5149; \quad \ddot{a}_x = 8.5693$. Using (4.13), $P_x^{(m)} = \frac{0.5149}{8.5693 - \frac{11}{24}} = 0.0635.$

### Comment

Note that this value for the annual premium payable monthly (0.0635) is higher than the annual payment obtained in (4.12) of a value of 0.0600. This is due to the value earned by the insurance company from a higher frequency of compounding.

This chapter and eBook are concluded.

# 5 REFERENCES

**Selected References**

All references herein are valuable for the study of life insurance; but it is our professional opinion that three warrant special mention for their respective comprehensive coverage of issues therein. These are:

David C.M. Dickson, Mary R. Hardy and Howard R. Waters, *Actuarial Mathematics for Life Contingent Risks,* Cambridge University Press (2009).

Hans U. Gerber, *Life Insurance Mathematics,* Third Edition, Springer, 1997.

Annamaria Olivieri and Ermanno Pitacco (2010). Introduction to Insurance Mathematics, Second Edition, Springer.

**The remaining important references are as follows:**

Anousschka van der Meulen, *Life Tables and Survival Analysis*, Statistics Netherlands, The Hague, 2012.

Eric V. Slud, *Actuarial Mathematics and Life Table Statistics,* Mathematics Department, University of Maryland, College Park (2001).

Francis S. Perryman, *International Actuarial Notation,* Proceedings of the Casualty Actuarial Society, Vol. 36, No. 66, 1949.

**Web Resources:**

Jerry A. Veeh, *Lecture Notes on Actuarial Mathematics,* 2006.
http://javeeh.net/lecnotes/actmath.pdf.

Ragnar Norberg, *Basic Life Insurance Mathematics* (2002)
www.math.ku.dk/~mogens/lifebook.pdf

Herman Pat Goeters, *Annuities, Insurance and Life*, Department of Mathematics Auburn University Auburn, Alabama, USA
http://www.auburn.edu/~goetehp/acttex/acttex.pdf